

# EXCEPTIONAL ZEROS OF $L$ -SERIES AND BERNOULLI-CARLITZ NUMBERS

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ABSTRACT. Bernoulli-Carlitz numbers were introduced by L. Carlitz in 1935, they are the analogues in positive characteristic of Bernoulli numbers. We prove a conjecture formulated by F. Pellarin and the first author on the non-vanishing modulo a given prime of families of Bernoulli-Carlitz numbers. We then show that the "exceptional zeros" of certain  $L$ -series are intimately connected to the Bernoulli-Carlitz numbers.

With an appendix by B. Anglès, D. Goss, F. Pellarin, F. Tavares Ribeiro

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## 1. INTRODUCTION

Recently, M. Kaneko and D. Zagier have introduced the  $\mathbb{Q}$ -algebra of finite multiple zeta values which is a sub- $\mathbb{Q}$ -algebra of  $\mathcal{A} := \frac{\prod_p \mathbb{F}_p}{\oplus_p \mathbb{F}_p}$  ( $p$  runs through the prime numbers). This algebra of finite multiple zeta values contains the following elements:

$$\forall k \geq 2, \mathcal{Z}(k) = \left( \left( \frac{B_{p-k}}{k} \right)_p \right) \in \mathcal{A},$$

where  $B_n$  denotes the  $n$ th Bernoulli number. It is not known that the algebra of finite multiple zeta values is non-trivial. In particular, it is an open problem to prove that  $\mathcal{Z}(k) \neq 0$  for  $k \geq 3, k \equiv 1 \pmod{2}$  (observe that  $\mathcal{Z}(k) = 0$  if  $k \geq 2, k \equiv 0 \pmod{2}$ ). This latter problem is equivalent to the following:

**Conjecture 1.** *Let  $k \geq 3$  be an odd integer. Then, there exist infinitely many primes  $p$  such that  $B_{p-k} \not\equiv 0 \pmod{p}$ .*

Let  $k \geq 3$  be an odd integer. M. Kaneko ([14]) remarked that, viewing the  $B_{p-k}$ 's as being random modulo  $p$  when  $p$  varies through the prime numbers, taking into account that  $\sum_p \frac{1}{p}$  diverges, then it is reasonable to expect that there exist infinitely many prime numbers  $p$  such that  $B_{p-k} \equiv 0 \pmod{p}$ .

Let  $\mathbb{F}_q$  be a finite field having  $q$  elements,  $q$  being a power of a prime number  $p$ , and let  $\theta$  be an indeterminate over  $\mathbb{F}_q$ . In 1935, L. Carlitz has introduced the analogues of Bernoulli numbers for  $A := \mathbb{F}_q[\theta]$  ([9]). The Bernoulli-Carlitz numbers,  $BC_n \in K := \mathbb{F}_q(\theta)$ ,  $n \in \mathbb{N}$ , are defined as follows:

- $BC_n = 0$  if  $n \not\equiv 0 \pmod{q-1}$ ,
- for  $n \equiv 0 \pmod{q-1}$ , we have:

$$\frac{BC_n}{\Pi(n)} = \frac{\zeta_A(n)}{\tilde{\pi}^n},$$

where  $\Pi(n) \in A$  is the Carlitz factorial ([13], chapter 9, paragraph 9.1),  $\tilde{\pi}$  is the Carlitz period ([13], chapter 3, paragraph 3.2), and  $\zeta_A(n) := \sum_{a \in A, a \text{ monic}} \frac{1}{a^n} \in K_\infty := \mathbb{F}_p((\frac{1}{\theta}))$  is the value at  $n$  of the Carlitz-Goss zeta function. The Bernoulli-Carlitz numbers are connected to Taelman's class modules introduced in [19] (see for example [21] and [6]). L. Carlitz established a von-Staudt result for these numbers ([13], chapter 9, paragraph 9.2), and as an easy consequence, we get that if  $P$  is a monic irreducible polynomial in  $A$ , then  $BC_n$  is  $P$ -integral for  $0 \leq n \leq q^{\deg_\theta P} - 2$ . It is natural to ask if Conjecture 1 is valid in the carlitzian context. In this paper, we prove a stronger result which answers positively to a Conjecture formulated in [3]:

**Theorem 1.1.** *Let  $N \geq 2$  be an integer,  $N \equiv 1 \pmod{q-1}$ . Let  $\ell_q(N)$  be the sum of the digits in base  $q$  of  $N$ . Let  $P \in A$  be a monic irreducible polynomial of degree  $d$  such that  $q^d > N$ . If  $d \geq \frac{\ell_q(N)-1}{q-1}N$ , then:*

$$BC_{q^d-N} \not\equiv 0 \pmod{P}.$$

The above Theorem is linked with the study of exceptional zeros of certain  $L$ -series introduced in 2012 by F. Pellarin ([15]), but from a slightly different point of view. More precisely, let  $N$  be as above and for simplicity we assume that  $\ell_q(N) \geq q$ , let  $t$  be an indeterminate over  $K_\infty$ , let's consider:

$$\mathcal{L}_N(t) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A[[\frac{1}{t}]]^\times,$$

where  $A_{+,d}$  is the set of monic elements in  $A$  of degree  $d$ . It was already noticed by F. Pellarin ([16]) that such  $L$ -series can be related with Anderson's solitons and should play an important role in the arithmetic theory of function fields. Let  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure of  $K_\infty$ . Then, one can show that  $\mathcal{L}_N(t)$  converges on  $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\}$ , where  $v_\infty$  is the valuation on  $\mathbb{C}_\infty$  normalized such that  $v_\infty(\theta) = -1$ . Furthermore, one can easily see that the elements of  $S := \{\theta^{q^j}, j \in \mathbb{Z}, q^j \leq N\}$  are zeros of the function  $\mathcal{L}_N(t)$ . We call the zeros of  $\mathcal{L}_N(t)$  which belong to  $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\} \setminus S$  the *exceptional zeros* of  $\mathcal{L}_N(t)$ . Let's briefly describe the case  $q = p$ . In this case, the exceptional zeros of  $\mathcal{L}_N(t)$  are simple, belong to  $\mathbb{F}_p((\frac{1}{\theta}))$  and are the eigenvalues of a certain  $K$ -linear

endomorphism  $\phi_t^{(N)}$  of a finite dimensional  $K$ -vector space  $H(\phi^{(N)})$  connected to the generalization of Taelman's class modules introduced in [5]. The proof of the fact that the exceptional zeros are simple and "real" uses combinatorial techniques introduced by F. Diaz-Vargas ([12]) and J. Sheats ([18]). Furthermore, if  $p^d > N$ , then:

$$BC_{p^d-N} \frac{(-1)^{\frac{\ell_p(N)-p}{p-1}} \prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (\theta^{p^l} - \theta^{p^n})^{n_l}}{\Pi(N)\Pi(p^d-N)} = \det_K (\theta^{p^d} \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}),$$

where  $\Pi(\cdot)$  is the Carlitz factorial, and  $N = \sum_{l=0}^k n_l p^l$ ,  $n_l \in \{0, \dots, p-1\}$ . Since the eigenvalues of  $\phi_t^{(N)}$  are exactly in this situation the exceptional zeros of  $\mathcal{L}_N(t)$ , we also obtain another proof of Theorem 1.1 as a consequence of the fact that:

$$\det_{K[Z]} (Z \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \in \mathbb{F}_p[Z, \theta],$$

and therefore ( $P$  is a monic irreducible polynomial of degree  $d$ ):

$$\det_K (\theta^{p^d} \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \equiv \det_K (\theta \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \pmod{P}.$$

Let's observe that Theorem 1.1 implies the following (see [3], page 248):

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{(a')^N}{a} \neq 0,$$

where  $a'$  denotes the derivative of  $a$  and  $N \equiv 1 \pmod{q-1}$ . In the appendix of this paper, we discuss a digit principle for such Euler type sums.

We mention that the construction of Kaneko-Zagier's objects in the positive characteristic world is the subject of a forthcoming work of F. Pellarin and R. Perkins ([17]), they prove, in this context, that the algebra of finite multiple zeta values is non-trivial. In this situation, it would be very interesting to examine the validity of Conjecture 1 for Bernoulli-Goss numbers (see [2] for a special case).

## 2. PROOF OF THEOREM 1.1

**2.1. Notation.** Let  $\mathbb{F}_q$  be a finite field having  $q$  elements and let  $p$  be the characteristic of  $\mathbb{F}_q$ . Let  $\theta$  be an indeterminate over  $\mathbb{F}_q$  and let  $A = \mathbb{F}_q[\theta]$ ,  $K = \mathbb{F}_q(\theta)$ ,  $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ . Let  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure of  $K_\infty$ . Let  $v_\infty : \mathbb{C}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\}$  be the valuation on  $\mathbb{C}_\infty$  normalized such that  $v_\infty(\theta) = -1$ . For  $d \in \mathbb{N}$ , let  $A_{+,d}$  be the set of monic elements in  $A$  of degree  $d$ .

### 2.2. The $L$ -series $L_N(t)$ .

Let  $N \geq 1$  be an integer. Let  $t$  be an indeterminate over  $\mathbb{C}_\infty$ . Let  $\mathbb{T}_t$  be the Tate algebra in the variable  $t$  with coefficients in  $\mathbb{C}_\infty$ . Let's set:

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{T}_t^\times.$$

Then, we can write:

$$L_N(t) = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \quad \alpha_{i,N}(t) \in \mathbb{F}_q[t].$$

Observe that  $\alpha_{0,N}(t) = 1$ .

**Lemma 2.1.** *We have:*

$$\forall i \geq 0, \deg_t \alpha_{i,N}(t) \leq N(\text{Max}\{\frac{\log(i)}{\log(q)}, 0\} + [\frac{\ell_q(N)}{q-1}] + 1).$$

*In particular  $L_N(t)$  is an entire function.*

*Proof.* Let  $u = [\frac{\ell_q(N)}{q-1}] \in \mathbb{N}$ . This Lemma is a consequence of the proof of [3], Lemma 7. We give a proof for the convenience of the reader. We will use the following elementary fact ([3], Lemma 4):

Let  $s \geq 1$  be an integer and let  $t_1, \dots, t_s$  be  $s$  indeterminate over  $\mathbb{F}_p$ . If  $d(q-1) > s$  then  $\sum_{a \in A_{+,d}} a(t_1) \cdots a(t_s) = 0$ .

If  $a$  is a monic polynomial in  $A$ , we will set:

$$\langle a \rangle_\infty = \frac{a}{\theta^{\deg_\theta a}} \in 1 + \frac{1}{\theta} \mathbb{F}_q[[\frac{1}{\theta}]].$$

Let  $S_d := \sum_{a \in A_{+,d}} \frac{a(t)^N}{a}$ . Observe that:

$$\deg_t S_d = dN.$$

We have :

$$S_d = \frac{1}{\theta^d} \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1}.$$

Observe that ( $p$ -adically)  $-1 = \sum_{n \geq 0} (q-1)q^n$ . For  $m \geq 0$ , set:

$$y_m = \sum_{n=0}^m (q-1)q^n.$$

Then:

$$y_m \equiv -1 \pmod{q^{m+1}}, \quad \ell_q(y_m) = (m+1)(q-1).$$

Therefore:

$$v_\infty(\sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1} - \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{y_m}) \geq q^{m+1},$$

where  $v_\infty$  is the  $\infty$ -adic valuation on  $\mathbb{T}_t$  such that  $v_\infty(\theta) = -1$ . Thus, if  $\ell_q(N) + (m+1)(q-1) < d(q-1)$ , we get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{y_m} = 0.$$

We therefore get, if  $d \geq u+2$  :

$$v_\infty(S_d) \geq d + q^{d-u-1}.$$

This implies that  $L_N(t)$  is an entire function. Let  $j$  such that  $t^j$  appears in  $\alpha_{i,N}(t)$ . Let  $x = [\frac{j}{N}]$ . Let  $d$  be minimal such that  $t^j$  comes from  $S_d$ . We must have  $d \geq x$  and  $j \leq dN$ . Furthermore, if  $d \geq u+2$ , we have:

$$i \geq d + q^{d-u-1}.$$

In particular:

$$i \geq d + q^{d-u-1} \geq q^{d-u-1}.$$

Therefore:

$$d \leq \text{Max}\{\frac{\log(i)}{\log(q)}, 0\} + u + 1.$$

□

### 2.3. The two variable polynomial $B_N(t, \theta)$ .

Let  $N \geq 2$ ,  $N \equiv 1 \pmod{q-1}$ . If  $\ell_q(N) = 1$ , we set  $B_N(t, \theta) = 1$ . Let's assume that  $\ell_q(N) \neq 1$  and let's set  $s = \ell_q(N) \geq 2$ . Let  $t_1, \dots, t_s$  be  $s$  indeterminates over  $\mathbb{C}_\infty$ . Let  $\mathbb{T}_s$  be the Tate algebra in the indeterminates  $t_1, \dots, t_s$  with coefficients in  $\mathbb{C}_\infty$ . Let  $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$  be the continuous morphism of  $\mathbb{F}_q[t_1, \dots, t_s]$ -algebras such that  $\forall c \in \mathbb{C}_\infty, \tau(c) = c^q$ . For  $i = 1, \dots, s$ , we set:

$$\omega(t_i) = \lambda_\theta \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_s,$$

where  $\lambda_\theta$  is a fixed  $(q-1)$ th-root of  $-\theta$  in  $\mathbb{C}_\infty$ . Set:

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1}.$$

Set:

$$L_s = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We also set:

$$\mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{L_s \omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}} \in \mathbb{T}_s.$$

Then, by [5], Lemma 7.6 (see also [3], Corollary 21),  $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$  is a monic polynomial in  $\theta$  of degree  $r = \frac{s-q}{q-1}$ . Write  $N = \sum_{n=1}^{\ell_q(N)} q^{e_n}$ ,  $e_1 \leq e_2 \leq \dots \leq e_{\ell_q(N)}$ . We set:

$$B_N(t, \theta) = \mathbb{B}_s|_{t_i = t^{q^{e_i}}} \in \mathbb{F}_q[t, \theta].$$

We observe that  $B_N(t, \theta)$  is a monic polynomial in  $\theta$  such that  $\deg_\theta B_N(t, \theta) = r$ .

**Lemma 2.2.** *Let  $N \geq 2$ ,  $N \equiv 1 \pmod{q-1}$ . Then:*

- 1)  $B_N(t^p, \theta^p) = B_N(t, \theta)^p$ .
- 2)  $B_{qN}(t, \theta) = B_N(t^q, \theta)$ .
- 3) *We have:*

$$B_N(t, \theta) \equiv (\theta - t)^r - r(t^q - t)(\theta - t)^{r-1} \pmod{(t^q - t)^2 \mathbb{F}_p[t, \theta]}.$$

- 4) *If  $N \equiv 0 \pmod{p}$ , then  $B_N(t, \theta) \in \mathbb{F}_p[t^p, \theta]$ .*

*Proof.* Recall that:

$$B_N(t, \theta) = \frac{(-1)^{\frac{\ell_q(N)-1}{q-1}}}{\tilde{\pi}} \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \prod_{l=0}^k \omega(t^{q^l})^{n_l},$$

where  $N = \sum_{l=0}^k n_l q^l$ ,  $n_0, \dots, n_k \in \{0, \dots, q-1\}$ . Observe that:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{F}_p[t][[\frac{1}{\theta}]].$$

Thus:

$$B_N(t, \theta) \in \mathbb{F}_p[t, \theta].$$

Thus we get assertion 1). Assertion 2) is a consequence of the definition of  $B_N(t, \theta)$ .

Let  $\zeta \in \mathbb{F}_q$ . By [4], theorem 2.9, we have:

$$\omega(t)|_{t=\zeta} = \exp_C\left(\frac{\tilde{\pi}}{\theta - \zeta}\right),$$

where  $\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  is the Carlitz exponential ([13], chapter 3, paragraph 3.2). Now, by [15] Theorem 1, we get:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \Big|_{t=\zeta} = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(\zeta)}{a} = \frac{\tilde{\pi}}{(\theta - \zeta) \exp_C(\frac{\tilde{\pi}}{\theta - \zeta})}.$$

But observe that:

$$\exp_C(\frac{\tilde{\pi}}{\theta - \zeta})^{q-1} = -(\theta - \zeta).$$

Thus:

$$B_N(t, \theta) \Big|_{t=\zeta} = (\theta - \zeta)^r.$$

Observe that  $\frac{d}{dt} B_N(t, \theta)$  is equal to:

$$\frac{n_0(-1)^{r+1}}{\tilde{\pi}} \left( \prod_{l=0}^k \omega(t^{q^l})^{-n_l} \right) \sum_{d \geq 0} \sum_{a \in A_{+,d}} \left( \frac{\frac{d}{dt}(a(t))a(t)^{N-1}}{a} - \frac{a(t)^N}{a} \frac{\frac{d}{dt}(\omega(t))}{\omega(t)} \right).$$

Thus we get assertion 4). Since for  $\ell_q(N) = q$ , we have  $B_N(t) = 1$ . We get:

$$\forall \zeta \in \mathbb{F}_q, \frac{d}{dt} B_N(t, \theta) \Big|_{t=\zeta} = 0.$$

This concludes the proof of the Lemma.  $\square$

**Lemma 2.3.** *Let  $N \geq 2, N \equiv 1 \pmod{q-1}$ . Then  $\deg_t B_N(t, \theta) \geq p$  if  $r \geq 1$  and the total degree in  $t, \theta$  of  $B_N(t, \theta)$  is less than or equal to  $\text{Max}\{rN + r - 2, 0\}$ . Furthermore  $B_N(t, \theta)$  (as a polynomial in  $t$ ) is a primitive polynomial.*

*Proof.* Recall that if  $r = 0$  then  $B_N(t, \theta) = 1$ . Let's assume that  $r \geq 1$ . Observe that by Lemma 2.2, we have:

$$B_N(t, 0) \equiv -(-t)^{r-1}(t + r(t^q - t)) \pmod{(t^q - t)^2 \mathbb{F}_p[t]}.$$

In particular  $\deg_t B_N(t, \theta) \geq p$ . Let  $x \in \mathbb{C}_\infty$  such that  $v_\infty(x) > \frac{-1}{N}$ . Then:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(x)^N}{a} = \prod_{P \text{ monic irreducible in } A} \left(1 - \frac{P(x)^N}{P}\right)^{-1} \in \mathbb{C}_\infty^\times.$$

Write  $N = \sum_{l=0}^k n_l q^l, n_l \in \{0, \dots, q-1\}, n_k \neq 0$ . For  $l = 0, \dots, k$ , we have:

$$\omega(t^{q^l}) \Big|_{t=x} \in \mathbb{C}_\infty^\times.$$

Therefore:

$$B_N(t, \theta) \Big|_{t=x} \neq 0.$$

This implies that, if  $x \in \mathbb{C}_\infty$  is a root of  $B_N(t, \theta)$  then  $v_\infty(x) \leq \frac{-1}{N} < 0$ . Write in  $\mathbb{C}_\infty[t]$ :

$$B_N(t, \theta) = \lambda \prod_{j=1}^m (t - x_j), \lambda \in \mathbb{F}_p[\theta] \setminus \{0\}, \deg_\theta \lambda \leq r-1, x_1, \dots, x_m \in \mathbb{C}_\infty,$$

where  $m = \deg_t B_N(t, \theta)$ . Then:

$$\theta^r = (-1)^m \lambda \prod_{j=1}^m x_j.$$

Therefore:

$$\deg_{\theta} \lambda - r \leq \frac{-m}{N}.$$

We finally get:

$$\deg_t B_N(t, \theta) \leq (r - \deg_{\theta} \lambda)N \leq rN.$$

Since  $B_N(t, \theta)$  is a monic polynomial in  $\theta$ , the total degree in  $t, \theta$  of  $B_N(t, \theta)$  is less than or equal to  $\deg_t B_N(t, \theta) + r - 2$ .

Write:

$$B_N(t, \theta) = \alpha F, \alpha \in \mathbb{F}_p[\theta] \setminus \{0\},$$

where  $F$  is a primitive polynomial (as a polynomial in  $t$ ). In particular  $\alpha$  must divide  $\theta^r$  and  $B_N(1, \theta)$  in  $\mathbb{F}_p[\theta]$ . By Lemma 2.2, this implies that  $\alpha \in \mathbb{F}_p^{\times}$ .  $\square$

**Remark 2.4.** Let  $f(\theta)$  be a monic irreducible polynomial in  $\mathbb{F}_p[\theta]$ . Let  $P_1, \dots, P_m$  be the monic irreducible polynomials in  $A$  such that  $f(\theta) = P_1 \cdots P_m$ . We can order them such that if  $d = \deg_{\theta} f$ , then for  $i = 1, \dots, m$ , we have:

$$P_i = \sigma^{i-1}(P_1),$$

where  $\sigma : A \rightarrow A$  is the morphism of  $\mathbb{F}_p[\theta]$ -algebras such that  $\forall x \in \mathbb{F}_q$ ,  $\sigma(x) = x^p$ , and  $m = [\mathbb{F}_{p^d} \cap \mathbb{F}_q : \mathbb{F}_p]$ . Let  $N \geq 1$ , and let's set:

$$\chi_N(f) = \prod_{i=1}^m (P_i(\theta) - P_i(t)^N) - f(\theta) \in \mathbb{F}_p[t, \theta].$$

We have:

$$\deg_t \chi_N(f) = N \deg_{\theta} f,$$

$$\deg_{\theta} \chi_N(f) \leq \deg_{\theta} f - \deg_{\theta} P_1 \leq \deg_{\theta}(f) \left(1 - \frac{\log(p)}{\log(q)}\right).$$

If  $f_1, \dots, f_n$  are  $n$  irreducible monic polynomials in  $\mathbb{F}_p[\theta]$ , we set:

$$\chi_N(f_1 \cdots f_m) = \prod_{l=1}^n \chi_N(f_l) \in \mathbb{F}_p[t, \theta].$$

Thus in  $\mathbb{F}_p[t][\left(\frac{1}{\theta}\right)]$ :

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a}.$$

Observe that  $\sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a}$  converges on  $\{x \in \mathbb{C}_{\infty}, v_{\infty}(x) > \frac{-1}{N}\}$  and does not vanish. Therefore on  $\{x \in \mathbb{C}_{\infty}, v_{\infty}(x) > \frac{-1}{N}\}$  :

$$\sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a} = \frac{1}{\theta^{\frac{\ell_q(N)-q}{q-1}}} B_N(t, \theta) \prod_{l=0}^k \prod_{j \geq 0} \left(1 - \frac{t^{q^l}}{\theta^{q^j}}\right)^{n_l},$$

where  $N \equiv 1 \pmod{q-1}$ ,  $\ell_q(N) \geq q$ ,  $N = \sum_{l=0}^k n_l q^l$ ,  $n_l \in \{0, \dots, q-1\}$ ,  $n_k \neq 0$ .

Let  $s \geq 2$ ,  $s \equiv 1 \pmod{q-1}$ . Recall that we have set:

$$\mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{L_s \omega(t_1) \cdots \omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s,$$

where  $\mathbb{T}_s$  is the Tate algebra in the indeterminates  $t_1, \dots, t_s$  with coefficients in  $\mathbb{C}_{\infty}$ , and for  $i = 1, \dots, s$ ,  $\omega(t_i) = \lambda_{\theta} \prod_{j \geq 0} (1 - \frac{t_i}{\theta^{q^j}})^{-1}$ . For  $m \in \mathbb{N}$ , we denote by  $BC_m \in K$  the  $m$ th Bernoulli-Carlitz number ([13], chapter 9, paragraph 9.2).

**Proposition 2.5.**

1) Let  $N \geq 1, N \equiv 1 \pmod{q-1}$ ,  $\ell_q(N) \geq q$ . Recall that  $r = \frac{\ell_q(N)-q}{q-1}$ . Let  $d \geq 1$  such that  $q^d > N$ , then we have the following equality in  $\mathbb{C}_\infty$  :

$$\frac{B_N(\theta, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (\theta^{q^l} - \theta^{q^n})^{n_l}} = (-1)^r \frac{BC_{q^d-N}}{\Pi(N)\Pi(q^d-N)},$$

where  $\Pi(\cdot)$  is the Carlitz factorial, and  $N = \sum_{l=0}^k n_l q^l, n_l \in \{0, \dots, q-1\}$ .

2) Let  $N \geq 2, N \equiv 1 \pmod{q-1}$ . Let  $P$  be a monic irreducible polynomial in  $A$  of degree  $d \geq 1$  such that  $q^d > N$ . Then  $BC_{q^d-N} \equiv 0 \pmod{P}$  if and only if  $B_N(\theta, \theta) \equiv 0 \pmod{P}$ .

*Proof.*

1) The first assertion of the Proposition is a consequence of the proof of Theorem 2 in [3]. For the convenience of the reader, we give the proof of this result. Let  $s \geq q$ ,  $s \equiv 1 \pmod{q-1}$ . Then ([5], Lemma 7.6), we have that  $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$  is a monic polynomial in  $\theta$  of degree  $\frac{s-q}{q-1}$ .

Let  $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$  be the continuous morphism of  $\mathbb{F}_q[t_1, \dots, t_s]$ -algebras such that  $\forall x \in \mathbb{C}_\infty, \tau(x) = x^q$ . Since  $\lambda_\theta^q = -\theta \lambda_\theta$ , we have:

$$\tau(\omega(t_i)) = (t_i - \theta)\omega(t_i), i = 1, \dots, s.$$

Let  $d \geq 1$ , we get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s)}{\tilde{\pi}^{q^d}} \omega(t_1) \dots \omega(t_s) = \frac{\tau^d(\mathbb{B}_s)}{\prod_{l=0}^{d-1} (t_1 - \theta^{q^l}) \dots (t_s - \theta^{q^l})}.$$

Recall that, by formula (24) in [15], we have:

$$(t_i - \theta^{q^l})\omega(t_i) \big|_{t=\theta^{q^l}} = -\frac{\tilde{\pi}^{q^l}}{D_l}.$$

Let  $l_1, \dots, l_s \in \mathbb{N}$ , we get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s)}{\tilde{\pi}^{q^d}} (t_1 - \theta^{q^{l_1}})\omega(t_1) \dots (t_s - \theta^{q^{l_s}})\omega(t_s) = \frac{\tau^d(\mathbb{B}_s)(t_1 - \theta^{q^{l_1}}) \dots (t_s - \theta^{q^{l_s}})}{\prod_{l=0}^{d-1} (t_1 - \theta^{q^l}) \dots (t_s - \theta^{q^l})}.$$

Now, let  $N \geq 1$  such that  $\ell_q(N) = s$  (observe that  $N \equiv 1 \pmod{q-1}$  and  $\ell_q(N) \geq q$ ). Write  $N = \sum_{l=0}^k n_l q^l, n_0, \dots, n_k \in \{0, \dots, q-1\}, n_k \neq 0$ . Let  $d \geq k+1$ . We get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\sum_{u \geq 0} \sum_{a \in A_{+,u}} \frac{a(t)^N}{a^{q^d}}}{\tilde{\pi}^{q^d}} \left( \prod_{l=0}^k ((t^{q^l} - \theta^{q^l})\omega(t^{q^l}))^{n_l} \right) = \frac{B_N(t, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (t^{q^l} - \theta^{q^n})^{n_l}}.$$

We get:

$$\frac{B_N(t, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (t^{q^l} - \theta^{q^n})^{n_l}} \big|_{t=\theta} = (-1)^{\frac{\ell_q(N)-q}{q-1}} \frac{BC_{q^d-N}}{\Pi(N)\Pi(q^d-N)}.$$

2) The result is well-known for  $\ell_q(N) = 1$  (this is a consequence of the definition of the Bernoulli-Carlitz numbers and [13], Lemma 8.22.4). Thus, we will assume  $\ell_q(N) \geq q$ . The assertion is then a consequence of the fact that:

$$B_N(\theta, \theta) \equiv B_N(\theta, \theta^{q^d}) \pmod{P}.$$

□



We have already mentioned that  $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$  is a monic polynomial in  $\theta$  of degree  $\frac{s-q}{q-1}$  ([5], Lemma 7.6). Let's observe that we have:

**Lemma 2.6.** *For  $s \geq 2q - 1$ ,  $s \equiv 1 \pmod{q-1}$ , we have:*

$$\mathbb{B}_s(t_1, \dots, t_{s-(q-1)}, 0, \dots, 0) = (\theta - t_1 \cdots t_{s-(q-1)}) \mathbb{B}_{s-(q-1)}(t_1, \dots, t_{s-(q-1)}).$$

More generally, if  $\zeta$  is in the algebraic closure of  $\mathbb{F}_q$  in  $\mathbb{C}_\infty$ , let  $P$  be the monic irreducible polynomial in  $A$  such that  $P(\zeta) = 0$ . Let  $s \equiv 1 \pmod{q-1}$ ,  $s \geq q + q^d - 1$ , where  $d$  is the degree of  $P$ . Write  $s' = s - (q^d - 1)$ . We have:

$$\mathbb{B}_s(t_1, \dots, t_{s'}, \zeta, \dots, \zeta) = (P - P(t_1) \cdots P(t_{s'})) \mathbb{B}_{s'}(t_1, \dots, t_{s'}).$$

*Proof.* The polynomial  $\mathbb{B}_s(t_1, \dots, t_{s-(q-1)}, 0, \dots, 0)$  is equal to:

$$(-\theta)(-1)^{\frac{s-1}{q-1}} \frac{\omega(t_1) \cdots \omega(t_{s-(q-1)})}{\widetilde{\pi}} \prod_{\substack{P \text{ monic irreducible in } A, \\ P \neq \theta}} \left(1 - \frac{P(t_1) \cdots P(t_{s-(q-1)})}{P}\right)^{-1}.$$

The proof of the second assertion of the Lemma is similar, using [4], Theorem 2.9, and the properties of Gauss-Thakur sums ([22]).  $\square$

**Lemma 2.7.**

$$\mathbb{B}_{2q-1} = \theta - \sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}.$$

*Proof.* Let  $\mathbb{T}_{2q-1}(K_\infty)$  be the Tate algebra in the variable  $t_1, \dots, t_{2q-1}$  with coefficients in  $K_\infty$ . Then:

$$\mathbb{T}_{2q-1} = \frac{1}{\theta} A[t_1, \dots, t_{2q-1}] \oplus N,$$

where  $N = \{f \in \mathbb{T}_{2q-1}, v_\infty(f) \geq 2\}$ . Let  $\phi : A \rightarrow A[t_1, \dots, t_{2q-1}]\{\tau\}$  be the morphism of  $\mathbb{F}_q$ -algebras give by  $\phi_\theta = (t_1 - \theta) \cdots (t_{2q-1} - \theta)\tau + \theta$ . Then by the results in [5], we have:

$$N = \exp_\phi(\mathbb{T}_{2q-1}),$$

and  $H_\phi := \frac{\mathbb{T}_{2q-1}}{A[t_1, \dots, t_{2q-1}] \oplus N}$  is a free  $\mathbb{F}_q[t_1, \dots, t_{2q-1}]$ -module of rank one generated by  $\frac{1}{\theta}$ . Furthermore:

$$\mathbb{B}_s = \det_{\mathbb{F}_q[t_1, \dots, t_{2q-1}][Z]} (Z \text{Id} - \phi_\theta |_{H_\phi \otimes_{\mathbb{F}_q[t_1, \dots, t_{2q-1}]} \mathbb{F}_q[t_1, \dots, t_{2q-1}][Z]}) |_{Z=\theta}.$$

Now, observe that:

$$\phi_\theta\left(\frac{1}{\theta}\right) \equiv \frac{\sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}}{\theta} \pmod{A[t_1, \dots, t_{2q-1}] \oplus N}.$$

The Lemma follows.  $\square$

For  $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ , we set  $m_0 := s - (m_1 + \dots + m_d)$ , and:

$$\sigma_s(\underline{m}) = \sum \prod_{u=1}^d \prod_{i \in J_u} t_i^u,$$

where the sum runs through the disjoint unions  $J_1 \sqcup \dots \sqcup J_d \subset \{1, \dots, s\}$  such that  $|J_u| = m_u$ ,  $u = 1, \dots, d$ . Notice in particular that  $\sigma_s(\underline{m}) = 0$  if  $m_1 + \dots + m_d > s$ , that is, if  $m_0 < 0$ . To give an example, the above lemma shows that  $\mathbb{B}_{2q-1} = \theta - \sigma((q))$ .

**Lemma 2.8.** *Let  $\underline{m} \in \mathbb{N}^d$ . We have :*

$$\sigma_s(\underline{m}) \mid_{t_s=0, \dots, t_{s-(q-2)}=0} = \sigma_{s-(q-1)}(\underline{m}).$$

*In particular, if  $m_0 < q - 1$ , we have:*

$$\sigma_s(\underline{m}) \mid_{t_s=0, \dots, t_{s-(q-2)}=0} = 0.$$

*Proof.* This is a straight computation.  $\square$

Let  $\rho : \mathbb{F}_p[t_1, \dots, t_s] \rightarrow \mathbb{N} \cup \{+\infty\}$  be the function given by:  
- if  $f = 0$ ,  $\rho(f) = +\infty$ ,  
- if  $f \neq 0$ ,  $f = \sum \alpha_{j_1, \dots, j_s} t_1^{j_1} \cdots t_s^{j_s}$ ,  $\alpha_{j_1, \dots, j_s} \in \mathbb{F}_p$ , then  $\rho(f) = \inf\{j_1 + \dots + j_s, \alpha_{j_1, \dots, j_s} \neq 0\}$ .

Let's write:

$$\mathbb{B}_s = \sum_{i=0}^r B_{i,s} \theta^{r-i},$$

where  $B_{i,s} \in \mathbb{F}_p[t_1, \dots, t_s]$  is a symmetric polynomial, and  $B_{0,s} = 1$ .

**Proposition 2.9.** *For  $i = 1, \dots, r$ , we have:*

$$\rho(B_{i,s}) \geq i(q-1) + 1.$$

*Proof.* By Lemma 2.7, this is true for  $r = 1$ , thus we can assume that  $r \geq 2$ . The proof is by induction on  $r$ . Recall that by Lemma 2.2, we have:

$$\mathbb{B}_s \mid_{t_1=\dots=t_s=0} = \theta^r.$$

Thus, for  $i = 1, \dots, r$ , we can write:

$$B_{i,s} = \sum_{\underline{m} \in S} x_{i,\underline{m}} \sigma_s(\underline{m}), x_{i,\underline{m}} \in \mathbb{F}_p,$$

where  $S = \{(m_1, \dots, m_s) \in \mathbb{N}^s, 1 \leq m_1 + \dots + m_d \leq s\}$ . Set:

$$\widetilde{B}_{i,s} = B_{i,s} - \sum_{\underline{m} \in S, m_0 < q-1} x_{i,\underline{m}} \sigma_s(\underline{m}).$$

Then:

$$\rho(B_{i,s} - \widetilde{B}_{i,s}) \geq r(q-1) + 2.$$

Therefore we have to prove:

$$\rho(\widetilde{B}_{i,s}) \geq i(q-1) + 1.$$

Observe that, by Lemma 2.8, we have:

$$B_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = \widetilde{B}_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = \sum_{\underline{m} \in S, m_0 \geq q-1} x_{i,\underline{m}} \sigma_{s-(q-1)}(\underline{m}).$$

By Lemma 2.6, we have:

$$\mathbb{B}_s \mid_{t_s=\dots=t_{s-(q-2)}=0} = (\theta - t_1 \cdots t_{s-(q-1)}) \mathbb{B}_{s-(q-1)}.$$

We therefore get, for  $i = 1, \dots, r$ :

$$\widetilde{B}_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)},$$

where we have set  $B_{r,s-(q-1)} = 0$ . Now, by the induction hypothesis:

$$\rho(B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)}) \geq i(q-1) + 1.$$

Thus:

$$\rho(B_{i,s}) \geq i(q-1) + 1.$$

□

**Corollary 2.10.** *Let  $N \equiv 1 \pmod{q-1}$ ,  $\ell_q(N) \geq 2q-1$ . Then  $\forall a \in \overline{\mathbb{F}}_q[\theta]$ , we have:*

$$B_N(t, \theta) \big|_{t=a} \neq 0.$$

*Proof.* By Proposition 2.9, we have:

$$B_N(t, \theta) - \theta^r \in t(t, \theta)^r.$$

The Corollary follows easily. □

*Proof of Theorem 1.1*

We can assume that  $\ell_q(N) \geq q$ . By Lemma 2.3, the total degree in  $t, \theta$  of  $B_N(t, \theta)$  is strictly less than  $(r+1)N$ , where  $r = \frac{\ell_q(N)-q}{q-1}$ . Now, by Corollary 2.10:

$$B_N(\theta, \theta) \neq 0.$$

Furthermore,  $\deg_\theta B_N(\theta, \theta) < (r+1)N$ . Thus if  $P$  is a monic irreducible polynomial in  $A$  such that  $\deg_\theta P \geq (r+1)d$ , we have:

$$B_N(\theta, \theta) \not\equiv 0 \pmod{P}.$$

We conclude the proof of the Theorem by Proposition 2.5.

### 3. EXCEPTIONAL ZEROS AND EIGENVALUES OF CERTAIN $K$ -ENDOMORPHISMS

#### 3.1. The $L$ -series $L_N(t)$ .

Let  $N \geq 1$  be an integer. Recall that

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \quad \alpha_{i,N}(t) \in \mathbb{T}_t^\times.$$

Let  $d \geq 1$  be an integer, we set for  $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ :

- $\ell(\mathbf{k}) = d$ ,
- $|\mathbf{k}| = k_0 + \dots + k_{d-1}$ ,
- if  $a = a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1} + \theta^d$ ,  $a_i \in \mathbb{F}_q$ ,  $i = 0, \dots, d-1$ ,  $a^{\mathbf{k}} = \prod_{i=0}^{d-1} a_i^{k_i}$ .

Let's begin by a simple observation. Let  $d \geq 1$  and let  $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ . Let  $N \geq 1$  be an integer, we get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{a \in A_{+,d}} \sum_{\mathbf{m} \in \mathbb{N}^{d+1}, |\mathbf{m}|=N} C(N, \mathbf{m}) a^{\mathbf{k}} a^{\mathbf{m}} t^{m_1+2m_2+\dots+dm_d},$$

where:

$$C(N, \mathbf{m}) = \frac{N!}{m_0! \dots m_d!} \in \mathbb{F}_p.$$

Recall that by Luca's Theorem  $C(N, \mathbf{m}) \neq 0$  if and only if there is no carryover  $p$ -digits in the sum  $N = m_0 + \dots + m_d$ . Furthermore, recall that, for  $n \in \mathbb{N}$ ,  $\sum_{\lambda \in \mathbb{F}_q} \lambda^n \neq 0$  if and only if  $n \equiv 0 \pmod{q-1}$  and  $n \geq 1$ . Thus, for  $\mathbf{m} \in \mathbb{N}^{d+1}$ ,  $\sum_{a \in A_{+,d}} a^{\mathbf{m}} = 0$  unless  $(m_0, \dots, m_{d-1}) \in ((q-1)(\mathbb{N} \setminus \{0\}))^d$  and in this latter case  $\sum_{a \in A_{+,d}} a^{\mathbf{m}} = (-1)^d$ .

Thus for  $d, N \geq 1$ ,  $\mathbf{k} \in \{0, \dots, q-1\}^d$ , we denote by  $U_d(N, \mathbf{k})$  the set of elements  $\mathbf{m} \in \mathbb{N}^{d+1}$  such that:

- there is no carryover  $p$ -digits in the sum  $N = m_0 + \dots + m_d$ ,
- for  $n = 0, \dots, d-1$ ,  $m_n - k_n \in (q-1)\mathbb{N}$ .

For  $\mathbf{m} \in U_d(N, \mathbf{k})$ , we set:

$$\deg \mathbf{m} = m_1 + 2m_2 + \cdots + dm_d.$$

An element  $\mathbf{m} \in U_d(N, \mathbf{k})$  is called optimal if  $\deg \mathbf{m} = \text{Max}\{\deg \mathbf{n}, \mathbf{n} \in U_d(N, \mathbf{k})\}$ . If  $U_d(N, \mathbf{k}) \neq \emptyset$ , the greedy element of  $U_d(N, \mathbf{k})$  is the element  $\mathbf{m} = (m_0, \dots, m_d) \in U_d(N, \mathbf{k})$  such that  $(m_d, \dots, m_1)$  is largest lexicographically.

Let  $\mathbf{k} \in \mathbb{N}^d, d \geq 1$ . For  $n = 0, \dots, d-1$ , let  $\bar{k}_n \in \{0, \dots, q-1\}$  be the least integer such that  $k_n + \bar{k}_n \in (q-1)(\mathbb{N} \setminus \{0\})$ . We set:

$$\bar{\mathbf{k}} = (\bar{k}_0, \dots, \bar{k}_{d-1}).$$

We get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = (-1)^d \sum_{\mathbf{m} \in U_d(N, \bar{\mathbf{k}})} C(N, \mathbf{m}) t^{\deg \mathbf{m}}.$$

Let  $N \geq 1$  be an integer and let  $\ell_q(N)$  be the sum of digits of  $N$  in base  $q$ . Then we can write in a unique way:

$$N = \sum_{n=1}^{\ell_q(N)} q^{e_n}, e_1 \leq e_2 \leq \cdots \leq e_{\ell_q(N)}.$$

We set:

$$r = \text{Max}\{0, \lfloor \frac{\ell_q(N) - q}{q-1} \rfloor\} \in \mathbb{N}.$$

**Lemma 3.1.** *We have:*

$$\forall i \geq 0, \alpha_{i,N}(t) = \sum_{\ell(\mathbf{k}) + w(\mathbf{k}) = i} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \in \mathbb{F}_p[t],$$

where  $C_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{|\mathbf{k}|!}{k_0! \cdots k_{d-1}!} \in \mathbb{F}_p$ ,  $w(\mathbf{k}) = k_{d-1} + \cdots + (d-1)k_1 + dk_0$ , for  $\mathbf{k} = (k_0, \dots, k_{d-1})$ .

*Proof.* Let  $a \in A_{+,d}$ . We have:

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} C_{\mathbf{k}} a^{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}},$$

where  $C_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{|\mathbf{k}|!}{k_0! \cdots k_{d-1}!} \in \mathbb{F}_p$ ,  $w(\mathbf{k}) = k_{d-1} + \cdots + (d-1)k_1 + dk_0$ . Thus:

$$\sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \frac{(-1)^d}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} C_{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}} \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}}.$$

Therefore:

$$\alpha_{i,N}(t) = \sum_{\ell(\mathbf{k}) + w(\mathbf{k}) = i} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \in \mathbb{F}_p[t].$$

□

**Lemma 3.2.** *Let  $j \in \mathbb{Z}$ . Then  $L_N(t) \mid_{t=\theta q^j} = 0$  if and only if  $N \equiv 1 \pmod{q-1}$ , and  $q^j N > 1$ .*

*Proof.* This comes from the following facts:

-  $\forall n \geq 1, \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \neq 0$ ,

- for  $n \geq 0, \sum_{d \geq 0} \sum_{a \in A_{+,d}} a^n = 0$  if and only if  $n \geq 1, n \equiv 0 \pmod{q-1}$ . □

We will need the following Lemma in the sequel:

**Lemma 3.3.** *Let  $F(t) = \sum_{n \geq 0} \beta_n(t) \frac{1}{\theta^n} \in \mathbb{F}_q[t][[\frac{1}{\theta}]]$ ,  $\beta_n(t) \in \mathbb{F}_q[t]$ , and let  $\rho \in \mathbb{R}$  such that  $F(t)$  converges on  $\{x \in \mathbb{C}_\infty, v_\infty(x) \geq \rho\}$ . Let  $M \geq 1$  and set  $F_M(t) = \sum_{n=0}^M \beta_n(t) \frac{1}{\theta^n} \in K_\infty[t]$ . Let  $\varepsilon \in \mathbb{R}, \varepsilon \geq \rho$ . Suppose that  $F_M(t)$  has exactly  $k \geq 1$  zeros in  $\mathbb{C}_\infty$  with valuation  $\varepsilon$ . Then either  $F(t)$  has  $k$  zeros with valuation  $\varepsilon$  or  $F(t)$  has at least  $\deg_t F_M(t) + 1$  zeros with valuation  $> \varepsilon$ .*

*Proof.* Let's assume that the side of the Newton polygon of  $F_M(t)$  corresponding to the  $k$  zeros of valuation  $\varepsilon$  is not a portion of a side of the Newton polygon of  $F(t)$ , then  $F(t)$  has a side of slope  $-\varepsilon' < -\varepsilon$  with end point of abscissa  $k' > \deg_t F_M(t)$ . Thus the Newton polygon of  $F(t)$  delimited by the vertical axis of abscissas 0 and  $k'$  has only sides of slope  $\leq -\varepsilon'$ . Thus  $F(t)$  has  $k'$  zeros of valuation  $\geq \varepsilon'$ .  $\square$

### 3.2. An example.

For the convenience of the reader, we treat a basic example:  $N = 1$ . We set  $\ell_0 = 1$  and for  $d \geq 1$ ,  $\ell_d = (\theta - \theta^q) \cdots (\theta - \theta^{q^d})$ .

**Lemma 3.4.** *Let  $d \geq 0$ . Then:*

$$\sum_{a \in A_{+,d}} \frac{1}{a} = \frac{1}{\ell_d}.$$

*Proof.* This is a well-known consequence of a result of Carlitz ([13], Theorem 3.1.5). Let's give a proof for the convenience of the reader. We can assume that  $d \geq 1$ . Set:

$$e_d(X) = \prod_{a \in A, \deg_\theta a < d} (X - a) \in A[X].$$

Then ([13], Theorem 3.1.5):

$$e_d(X) = \sum_{i=0}^d \frac{D_d}{D_i \ell_{d-i}^{q^i}} X^{q^i},$$

where  $D_0 = 1$ , and for  $i \geq 1$ ,  $D_i = (\theta^{q^i} - \theta) D_{i-1}^q$ . Now observe that:

$$\frac{\frac{d}{dX}(e_d(X - \theta^d))}{e_d(X - \theta^d)} \Big|_{X=0} = - \sum_{a \in A_{+,d}} \frac{1}{a}.$$

Since  $e_d(\theta^d) = D_d$  ([13], Corollary 3.1.7), we get the desired result.  $\square$

**Lemma 3.5.** *Let  $d \geq 0$ . then:*

$$\sum_{a \in A_{+,d}} \frac{a(t)}{a} = \frac{(t - \theta) \cdots (t - \theta^{q^{d-1}})}{\ell_d}.$$

*Proof.* We can assume that  $d \geq 1$ . Set:

$$S = \sum_{a \in A_{+,d}} \frac{a(t)}{a}.$$

Then for  $i = 0, \dots, d-1$ , we have:

$$S \Big|_{t=\theta^{q^i}} = 0.$$

Furthermore, by Lemma 3.4,  $S$  has degree  $d$  in  $t$  and the coefficient of  $t^d$  is  $\frac{1}{\ell_d}$ . The Lemma follows.  $\square$

**Lemma 3.6.** *The edge points of the Newton polygon of  $L_1(t)$  are  $(d, q^{\frac{q^d-1}{q-1}})$ ,  $d \in \mathbb{N}$ .*

*Proof.* Let's write:

$$L_1(t) = \sum_{d \geq 0} S_d(t),$$

where:

$$S_d(t) = \sum_{a \in A_{+,d}} \frac{a(t)}{a}.$$

Let  $d \in \mathbb{N}$  and let  $d' > d$ . Let  $x \in K$  be the coefficient of  $t^d$  in  $S_{d'}(t)$ . Then, by Lemma 3.5, we get:

$$v_\infty(x) \geq -v_\infty(\ell_{d'}) - q^d - \dots - q^{d'-1} > q^{\frac{q^d-1}{q-1}}.$$

Thus, if we write:

$$L_1(t) = \sum_{d \geq 0} \alpha_d t^d \in K_\infty[[t]], \alpha_d \in K_\infty, d \in \mathbb{N},$$

by the above observation and again by Lemma 3.5, we get:

$$\forall d \geq 0, v_\infty(\alpha_d) = q^{\frac{q^d-1}{q-1}}.$$

□

This latter Lemma implies the following formula due to F. Pellarin ([15], Theorem 1):

**Proposition 3.7.** *Let  $\lambda_\theta \in \mathbb{C}_\infty^\times$  be a fixed  $(q-1)$ th root of  $-\theta$ . Set:*

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in \mathbb{C}_\infty^\times.$$

*Then:*

$$(\theta - t)L_1(t) = \frac{\tilde{\pi}}{\lambda_\theta} \prod_{j \geq 0} (1 - \frac{t}{\theta^{q^j}}).$$

*Proof.* We observe that:

$$\begin{aligned} \forall n \geq 1, L_1(t) |_{t=\theta^{q^n}} &= 0, \\ L_1(\theta) &= 1. \end{aligned}$$

By Lemma 3.6, the entire function  $(t - \theta)L_1(t)$  has simple zeros in  $K_\infty$  and if  $x \in K_\infty$  is such a zero,  $v_\infty(x) \in \{-q^i, i \in \mathbb{N}\}$ . Thus, there exists  $\alpha \in \mathbb{C}_\infty^\times$  such that:

$$(t - \theta)L_1(t) = \alpha \prod_{j \geq 0} (1 - \frac{t}{\theta^{q^j}}).$$

But, observe that:

$$\frac{\tilde{\pi}}{\lambda_\theta} \prod_{j \geq 1} (1 - \frac{t}{\theta^{q^j}}) |_{t=\theta} = \theta.$$

Therefore:

$$\alpha = \frac{-\tilde{\pi}}{\lambda_\theta}.$$

□

### 3.3. Eigenvalues and Bernoulli-Carlitz numbers.

In this paragraph, we slightly change our point of view. Let  $t$  be an indeterminate over  $\mathbb{C}_\infty$  and let  $\varphi : \mathbb{C}_\infty[[\frac{1}{t}]] \rightarrow \mathbb{C}_\infty[[\frac{1}{t}]]$  be the continuous (for the  $\frac{1}{t}$ -adic topology) morphism of  $\mathbb{C}_\infty$ -algebras such that  $\varphi(t) = t^q$ . We first recall some consequences of the work of F. Demeslay's (see the appendix of [5] or [11]) generalizing the work of L. Taelman ([20]).

Let  $N \geq 1, N \equiv 1 \pmod{q-1}, \ell_q(N) \geq q$ . Write  $N = \sum_{l=0}^k n_l q^l, n_l \in \{0, \dots, q-1\}, l = 0, \dots, k$ , and  $n_k \neq 0$ . We set  $B = K[t]$ . Let  $\phi^{(N)} : B \rightarrow B\{\varphi\}$  be the morphism of  $K$ -algebras given by:

$$\phi_t^{(N)} = \left( \prod_{l=0}^k (\theta^{q^l} - t)^{n_l} \right) \varphi + t.$$

Since  $t$  is transcendental over  $\mathbb{F}_q$ , there exists a unique "power series"  $\exp_{\phi^{(N)}} \in K(t)\{\{\varphi\}\}$  such that:

$$\begin{aligned} \exp_{\phi^{(N)}} &\equiv 1 \pmod{\varphi}, \\ \exp_{\phi^{(N)}} t &= \phi_t^{(N)} \exp_{\phi^{(N)}}. \end{aligned}$$

One can easily see that:

$$\exp_{\phi^{(N)}} = \sum_{j \geq 0} \frac{\prod_{l=0}^k (\prod_{n=0}^{j-1} (\theta^{q^l} - t^{q^n}))^{n_l}}{\prod_{n=0}^{j-1} (t^{q^n} - t^{q^j})} \varphi^j.$$

In particular  $\exp_{\phi^{(N)}}$  induces a continuous  $K$ -linear endomorphism of  $K((\frac{1}{t}))$  which is an isometry on a sufficiently small neighborhood of zero (for the  $\frac{1}{t}$ -adic topology). Let's set:

$$H(\phi^{(N)}) = \frac{K((\frac{1}{t}))}{(B + \exp_{\phi^{(N)}}(K((\frac{1}{t})))}.$$

Then  $H(\phi^{(N)})$  is a finite  $K$ -vector space and a  $B$ -module via  $\phi$ . Let's denote by  $[H(\phi^{(N)})]_B$  the monic generator (as a polynomial in  $t$ ) of the Fitting ideal of the  $B$ -module  $H(\phi^{(N)})$ , i.e.:

$$[H(\phi^{(N)})]_B = \det_{K[Z]} (Z \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) |_{Z=t}.$$

As in [5], Proposition 7.2, one can prove that:

$$\begin{aligned} \dim_K H(\phi^{(N)}) &= \frac{\ell_q(N) - q}{q - 1}, \\ \{x \in K((\frac{1}{t})), \exp_{\phi^{(N)}}(x) \in B\} &= \frac{\bar{\pi}}{t^{\frac{\ell_q(N)-q}{q-1}} \prod_{l=0}^k \bar{\omega}(\theta^{q^l})^{n_l}} B, \end{aligned}$$

where:

$$\begin{aligned} \bar{\pi} &= \prod_{j \geq 1} (1 - t^{1-q^j})^{-1} \in \mathbb{F}_p[[\frac{1}{t}]]^\times, \\ \bar{\omega}(\theta^{q^l}) &= \prod_{j \geq 0} (1 - \frac{\theta^{q^l}}{t^{q^j}})^{-1} \in A[[\frac{1}{t}]]^\times. \end{aligned}$$

Furthermore, if we set:

$$\mathcal{L}_N(t) = \prod_{P(t) \text{ monic irreducible polynomial of } \mathbb{F}_q[t]} \frac{[\frac{B}{P(t)B}]_B}{[\phi^{(N)}(\frac{B}{P(t)B})]_B},$$

then, by the appendix of [5],  $\mathcal{L}_N(t)$  converges in  $K((\frac{1}{t}))$ , and:

$$[H(\phi^{(N)})]_B \frac{\bar{\pi}}{t^{\frac{\ell_q(N)-q}{q-1}} \prod_{l=0}^k \bar{\omega}(\theta q^l)^{n_l}} = \mathcal{L}_N(t).$$

Now, one can compute  $\mathcal{L}_N(t)$  as in [5], paragraph 5.3, and we get:

$$\mathcal{L}_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A[[\frac{1}{t}]]^\times.$$

Therefore:

$$[H(\phi^{(N)})]_B = B_N(\theta, t).$$

We warn the reader not to confuse  $B_N(\theta, t)$  and  $B_N(t, \theta)$ , here and in the sequel of the paper, since we will be interested in those two polynomials. Recall that  $r = \frac{\ell_q(N)-q}{q-1}$ . Let  $\alpha_1(N), \dots, \alpha_r(N) \in \mathbb{C}_\infty$  be the eigenvalues (counted with multiplicity) of the  $K$ -endomorphism of  $H(\phi^{(N)})$ :  $\phi_t^{(N)}$ . We get:

$$B_N(\theta, \theta) = \prod_{j=1}^r (\theta - \alpha_j(N)).$$

Recall that  $\mathcal{L}_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A[[\frac{1}{t}]]$ . By Lemma 2.1,  $\mathcal{L}_N(t)$  converges on  $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\}$ . Let's write  $N = \sum_{l=0}^k n_l q^l$ ,  $n_l \in \{0, \dots, q-1\}$ ,  $n_k \neq 0$ . Set  $S = \{\theta q^i, i \leq k\}$ . Then, by Lemma 3.2, the elements of  $S$  are zeros of  $\mathcal{L}_N(t)$ . The elements of  $S$  are called the trivial zeros of  $\mathcal{L}_N(t)$ . A zero of  $\mathcal{L}_N(t)$  which does belong to  $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\} \setminus S$  will be called an exceptional zero of  $\mathcal{L}_N(t)$ . It is clear that the exceptional zeros of  $\mathcal{L}_N(t)$  are roots of  $B_N(\theta, t)$  with the same multiplicity. Our aim in the remaining of the article is to study the following problem:

**Problem 1.**

*Let  $N \geq 2$ ,  $N \equiv 1 \pmod{q-1}$ ,  $\ell_q(N) \geq q$ . Then all the eigenvalues of  $\phi_t^{(N)}$  (viewed as a  $K$ -endomorphism of  $H(\phi^{(N)})$ ) are simple and belong to  $\mathbb{F}_p((\frac{1}{\theta}))$ .*

Theorem 1.1 implies that  $\theta$  is not an eigenvalue of  $\phi_t^{(N)}$ . We presently do not know whether another trivial zero of  $\mathcal{L}_N(t)$  can be an eigenvalue of  $\phi_t^{(N)}$ . On the other side, the above problem implies that the exceptional zeros of  $\mathcal{L}_N(t)$  are simple. Observe that, by Lemma 2.2, the above Problem has an affirmative answer for  $q \leq \ell_q(N) \leq 2q-1$ .

#### 4. ANSWER TO PROBLEM 1 FOR $q = p$

In this section we give an affirmative answer to Problem 1 in the case  $q = p$ . By Proposition 2.5 and Proposition 4.6 below, this implies Theorem 1.1. For the convenience of the reader, we have tried to keep the text of this section as self-contained as possible.

In this section  $q = p$ .



#### 4.1. Preliminaries.

Lemma 4.1 and Proposition 4.2 below are slight generalizations of the arguments used in the proof of Theorem 1 in [12].

**Lemma 4.1.** *Let  $d, N \geq 1$  and  $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \{0, \dots, p-1\}^d$ . We assume that  $|\mathbf{k}| \leq \ell_p(N)$ . For  $1 \leq i \leq d$ , we set  $\sigma_i = \sum_{n=0}^{i-1} k_n$ . We also set  $\sigma_0 = 0$  and  $\sigma_{d+1} = \ell_p(N)$ . Let  $\mathbf{m} = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$  be the element defined as follows:*

$$n = 0, \dots, d, \quad m_n = \sum_{l=\sigma_n+1}^{\sigma_{n+1}} p^{e_l}.$$

Then:

$$\mathbf{m} \in U_d(N, \mathbf{k}).$$

Furthermore  $\mathbf{m}$  is the greedy element of  $U_d(N, \mathbf{k})$ . In particular  $U_d(N, \mathbf{k}) \neq \emptyset$  if and only if  $|\mathbf{k}| \leq \ell_p(N)$ .

*Proof.* Observe that  $\sigma_d = |\mathbf{k}| \leq \ell_p(N)$ . Thus  $\mathbf{m}$  is well-defined. It is then straightforward to verify that  $\mathbf{m} \in U_d(N, \mathbf{k})$  and that  $\mathbf{m}$  is the greedy element of  $U_d(N, \mathbf{k})$ . Now assume that  $U_d(N, \mathbf{k}) \neq \emptyset$ . Let  $\mathbf{m}' \in U_d(N, \mathbf{k})$ . Then:

$$n = 0, \dots, d-1, \ell_p(m'_n) \equiv k_n \pmod{p-1}.$$

This implies:

$$n = 0, \dots, d-1, \ell_p(m'_n) \geq k_n.$$

Thus:

$$\ell_p(N) \geq \sum_{n=0}^{d-1} \ell_p(m'_n) \geq |\mathbf{k}|.$$

□

**Proposition 4.2.** *Let  $d, N \geq 1$  and  $\mathbf{k} \in \mathbb{N}^d$ . We assume that  $|\bar{\mathbf{k}}| \leq \ell_p(N)$ . Then  $U_d(N, \bar{\mathbf{k}})$  contains a unique optimal element which is equal to the greedy element of  $U_d(N, \bar{\mathbf{k}})$ . In particular  $\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \neq 0$  if and only if  $|\bar{\mathbf{k}}| \leq \ell_p(N)$ .*

*Proof.* Let  $\mathbf{u} = (u_0, \dots, u_d)$  be the greedy element of  $U_d(N, \bar{\mathbf{k}})$ . Let  $\mathbf{m} \in U_d(N, \bar{\mathbf{k}})$  such that  $\mathbf{m} \neq \mathbf{u}$ . We will show that  $\mathbf{m}$  is not optimal.

Write  $c_n = \ell_p(m_n)$ ,  $n = 0, \dots, d-1$ . Then :

$$n = 0, \dots, d-1, c_n \geq \bar{k}_n, c_n \equiv \bar{k}_n \pmod{p-1}.$$

For  $n = 0, \dots, d-1$ , there exist  $f_{n,1} \leq \dots \leq f_{n,c_n}$  such that we can write in a unique way:

$$m_n = \sum_{l=1}^{c_n} p^{f_{n,l}}.$$

Case 1) There exists an integer  $j$ ,  $0 \leq j \leq d-1$ , such that  $c_j > \bar{k}_j$ .

Let  $\mathbf{m}' \in \mathbb{N}^{d+1}$  be defined as follows:

-  $m'_n = m_n$  for  $0 \leq n \leq d-1$ ,  $n \neq j$ ,

-  $m'_j = \sum_{l=1}^{\bar{k}_j} p^{f_{j,l}}$ .

-  $m'_d = N - m'_0 - \dots - m'_{d-1} = m_d + m_j - m'_j$ .

Then  $\mathbf{m}' \in U_d(N, \mathbf{k})$  and:

$$\deg \mathbf{m}' = \deg \mathbf{m} + (d-j)(m_j - m'_j) > \deg \mathbf{m}.$$

Thus  $\mathbf{m}$  is not optimal.

Case 2) For  $n = 0, \dots, d-1$ ,  $c_n = \bar{k}_n$ .

Let  $j \in \{0, \dots, d-1\}$  be the smallest integer such that  $m_j \neq u_j$ . Then, by the construction of  $\mathbf{u}$ , we have:

$$m_j > u_j.$$

Thus there exists an integer  $l$  such that the number of times  $p^l$  appears in the sum of  $m_j$  as  $\bar{k}_j$  powers of  $p$  is strictly greater than the number of times it appears in the sum of  $u_j$  as  $\bar{k}_j$  powers of  $p$ . Also, there exists an integer  $v$  such that the number of times  $p^v$  appears in the sum of  $u_j$  as  $\bar{k}_j$  powers of  $p$  is strictly greater than the number of times it appears in the sum of  $m_j$  as  $\bar{k}_j$  powers of  $p$ . Thus there exists an integer  $t > j$  such that  $p^v$  appears in the sum of  $m_t$  as  $\ell_p(m_t)$  powers of  $p$ . We observe that, by the construction of  $\mathbf{u}$ , we can choose  $v$  and  $l$  such that  $v < l$ . Now set:

- for  $n = 0, \dots, d$ ,  $n \neq j, n \neq t$ ,  $m'_n = m_n$ ,
- $m'_j = m_j - p^l + p^v$ ,
- $m'_t = m_t - p^v + p^l$ .

Let  $\mathbf{m}' = (m'_0, \dots, m'_{d-1}) \in \mathbb{N}^d$ . Then  $\mathbf{m}' \in U_d(N, \bar{\mathbf{k}})$  and:

$$\deg \mathbf{m}' = \sum_{l=0}^d l m'_l = \deg \mathbf{m} + (t-j)(p^l - p^v) > \deg \mathbf{m}.$$

Thus  $\mathbf{m}$  is not optimal. □

We have the following key result:

**Proposition 4.3.** *Let  $d, N \geq 1$  and  $\mathbf{k} \in \mathbb{N}^d$ . We assume that  $\ell_p(N) \geq p$  and that  $d(p-1) \leq \ell_p(N) - p$ . Then:*

$$N(d-1) < \deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd.$$

*Proof.* It is clear that  $\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd$ . Observe that  $|\bar{\mathbf{k}}| \leq d(p-1)$ . Let  $\mathbf{m}$  be the greedy element of  $U_d(N, \bar{\mathbf{k}})$ . By Proposition 4.2, we have:

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{n=0}^d n m_n = dN - \sum_{n=1}^d n m_{d-n}.$$

By Lemma 4.1, we have:

$$n = 0, \dots, d-1, m_n \leq \bar{k}_n p^{e_{\bar{k}_0} + \dots + \bar{k}_n},$$

where we recall that:

$$N = \sum_{n=1}^{\ell_p(N)} p^{e_n}.$$

Let  $l = \ell_p(N) - 1$ . Observe that:

$$e_{\ell_p(N)-p-(p-1)t} \leq e_l - 1 - t.$$

Since  $d(p-1) \leq \ell_p(N) - p$ , we get:

$$n = 1, \dots, d, \bar{k}_0 + \dots + \bar{k}_{d-n} \leq (p-1)(d-n+1) \leq \ell_p(N) - p - (p-1)(n-1).$$

Thus:

$$n = 1, \dots, d, m_{d-n} \leq (p-1)p^{e_l - n},$$

Therefore:

$$\sum_{n=1}^d nm_{d-n} \leq (p-1)p^{e_l} \sum_{n=1}^d np^{-n}.$$

Recall that if  $x \in \mathbb{R} \setminus \{1\}$ , we have:

$$\sum_{n=1}^d nx^{n-1} = \frac{1 - x^{d+1} + (d+1)(x-1)x^d}{(x-1)^2}.$$

Thus:

$$(p-1) \sum_{n=1}^d np^{-n} = \frac{p - p^{-d} - (d+1)(p-1)p^{-d}}{p-1} < \frac{p}{p-1}.$$

Now:

$$\sum_{n=1}^d nm_{d-n} < \frac{p}{p-1} p^{e_l}.$$

Thus:

$$\sum_{n=1}^d nm_{d-n} < 2p^{e_l}.$$

But  $2p^{e_l} \leq N$  since  $l = \ell_p(N) - 1$ . Thus:

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = dN - \sum_{n=1}^d nm_{d-n} > dN - N.$$

□

#### 4.2. Newton polygons of truncated $L$ -series.

For  $i, j \geq 0$ , we set:

$$S_j(i) = \sum_{a \in A_{+,j}} a(t)^i \in \mathbb{F}_p[t].$$

Note that by Proposition 4.2, we have  $S_i(N) \neq 0$  for  $i = 0, \dots, r$ .

**Proposition 4.4.** *We have:*

$$i = 0, \dots, r, \deg_t \alpha_{i,N}(t) = \deg_t S_i(N).$$

*Proof.* We recall that:

$$r = \text{Max}\left\{\left\lfloor \frac{\ell_p(N) - p}{p-1} \right\rfloor, 0\right\} \in \mathbb{N}.$$

We can assume that  $r \geq 1$ . By Proposition 4.3, for  $i = 0, \dots, r$ ,

$$\text{Max}\left\{\deg_t \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}, w(\mathbf{k}) + \ell(\mathbf{k}) = i\right\}$$

is attained for a unique  $\mathbf{k}$  which is  $(0, \dots, 0) \in \mathbb{N}^i$ . It remains to apply Lemma 3.1. □

We set:

$$\Lambda_r(N) = \sum_{i=0}^r \alpha_i(N) \theta^{-i} \in K_\infty[t].$$

Let's write:

$$N = \sum_{l=0}^k n_l p^l, n_0, \dots, n_k \in \{0, \dots, p-1\}, n_k \neq 0.$$

**Proposition 4.5.** *We have  $\deg_t \Lambda_r(N) = \deg_t S_r(N)$ . Furthermore the edge points of the Newton polygon of  $\Lambda_r(N)$  are:*

$$(\deg_t S_j(N), j), j = 0, \dots, r.$$

*Proof.* We can assume that  $r \geq 1$ . By Proposition 4.2, we have  $U_r(N, (p-1, \dots, p-1)) \neq \emptyset$ . Let  $\mathbf{m} \in \mathbb{N}^{r+1}$  be the optimal element of  $U_r(N, (p-1, \dots, p-1))$  given by Proposition 4.2 and Lemma 4.1. For  $j = 0, \dots, r$ , let  $\mathbf{m}(j) = (m_0, \dots, m_{j-1}, N - \sum_{n=0}^{j-1} m_n) \in \mathbb{N}^{j+1}$ . Then, again by Proposition 4.2 and Lemma 4.1,  $\mathbf{m}(j)$  is the optimal element of  $U_j(N, (p-1, \dots, p-1))$ . Therefore,  $\deg_t S_j(N) = \deg \mathbf{m}(j)$ ,  $j = 0, \dots, r$ . For  $j = 0, \dots, r$ , we have:

$$p^{k+1} > N - \sum_{n=0}^{j-1} m_n > n_k p^k.$$

Now let  $j \in \{0, \dots, r-1\}$ , we have:

$$p^{k+1} > \deg \mathbf{m}(j+1) - \deg \mathbf{m}(j) = N - \sum_{n=0}^j m_n > n_k p^k.$$

Thus, by Proposition 4.4, we get  $\deg_t \Lambda_r(N) = \deg_t S_r(N)$ . Furthermore, we observe that for  $j \in \{1, \dots, r-1\}$ , we have:

$$\deg \mathbf{m}(j) - \deg \mathbf{m}(j-1) > \deg \mathbf{m}(j+1) - \deg \mathbf{m}(j).$$

Thus, one easily sees that the edge points of the Newton polygon of  $\Lambda_r(N)$  are  $(\deg \mathbf{m}(j), j)$ ,  $j = 0, \dots, r$ .  $\square$

#### 4.3. A positive answer to Problem 1.

Let  $N \geq 2$  be an integer,  $N \equiv 1 \pmod{p-1}$ . Recall that:

$$N = \sum_{l=0}^k n_l p^l, n_0, \dots, n_k \in \{0, \dots, p-1\}, n_k \neq 0.$$

Recall that  $\ell_p(N) = n_0 + \dots + n_k$  and  $r = \max\{\frac{\ell_p(N)-p}{p-1}, 0\}$ . Let  $b_N \in \mathbb{N}$  be the total degree in  $t, \theta$  of the polynomial  $B_N(t)$ .

**Proposition 4.6.** *The polynomial  $B_N(t, \theta)$  has only one monomial of total degree  $b_N$ , which is of the form  $t^{b_N}$ . Furthermore:*

$$b_N = \deg_t S_r(N).$$

*Proof.* We can assume that  $r \geq 1$ . First let's observe that:

$$\prod_{j \geq 1} (1 - \theta^{1-p^j})^{-1} B_N(t, \theta) \equiv (-1)^{\frac{\ell_p(N)-1}{p-1}} \delta_N \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]},$$

where:

$$\delta_N = \sum_{i=0}^r \alpha_{r-i}(N) \theta^i.$$

Let  $\varepsilon_N \in \mathbb{F}_p[t, \theta]$  be uniquely determined by the congruence:

$$\varepsilon_N \equiv \delta_N \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

Let  $i \in \{0, \dots, r\}$ . A monomial in the product

$$\alpha_i(N) \theta^i \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}$$

is of the form:

$$\theta^i t^j \frac{t^\alpha}{\theta^\beta}, j \leq \deg_t \alpha_i(t), \beta \leq i, \alpha \leq p^k \beta.$$

Thus the total degree of a monomial in

$$\alpha_i(N) \theta^i \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}$$

is less than or equal to  $p^k i + \deg_t \alpha_i(t)$ . To conclude the proof of the Proposition, we use the same arguments as that used in the proof of Proposition 4.5. By Proposition 4.2, we have  $U_r(N, (p-1, \dots, p-1)) \neq \emptyset$ . Let  $\mathbf{m} \in \mathbb{N}^{r+1}$  be the optimal element of  $U_r(N, (p-1, \dots, p-1))$  given by Proposition 4.2 and Lemma 4.1. For  $j = 0, \dots, r$ , let  $\mathbf{m}(j) = (m_0, \dots, m_{r-j-1}, N - \sum_{n=0}^{r-j-1} m_n) \in \mathbb{N}^{r-j+1}$ . Then, again by Proposition 4.2 and Lemma 4.1,  $\mathbf{m}(j)$  is the optimal element of  $U_{r-j}(N, (p-1, \dots, p-1))$ . For  $j = 0, \dots, r$ , we have:

$$N - \sum_{n=0}^{r-j-1} m_n > p^k.$$

Now let  $j \in \{0, \dots, r-1\}$ , we have:

$$\deg \mathbf{m}(j) - \deg \mathbf{m}(j+1) = N - \sum_{n=0}^{r-j-1} m_n > p^k.$$

Thus, by Proposition 4.3 and Proposition 4.4,  $\text{Max}\{p^k i + \deg_t \alpha_i(t), i = 0, \dots, r\}$  is attained exactly at  $i = 0$ . Again by Proposition 4.4, this implies that the total degree in  $t, \theta$  of  $\varepsilon_N$  is equal to  $\deg_t S_r(N)$  and that  $\varepsilon_N(t)$  has only one monomial of total degree  $\deg_t S_r(N)$  which is of the form  $t^{\deg_t S_r(N)}$ . The Proposition follows.  $\square$

**Theorem 4.7.** *Let  $N \geq 2, N \equiv 1 \pmod{p-1}$ . The polynomial  $B_N(\theta, t)$  (viewed as a polynomial in  $t$ ) has  $r$  simple roots and all its roots are contained in  $\mathbb{F}_p((\frac{1}{\theta})) \setminus \{\theta^{p^i}, i \in \mathbb{Z}\}$ .*

*Proof.*

Recall that  $B_N(t, \theta)$  is a monic polynomial in  $\theta$  such that  $\deg_\theta B_N(t, \theta) = r$ . We can assume that  $r \geq 1$ . By Proposition 4.6, the leading coefficient of  $B_N(t, \theta)$  as a polynomial in  $t$  is in  $\mathbb{F}_p^\times$  and:

$$b_N = \deg_t B_N(t, \theta) > r.$$

Let  $S = \{\theta^{p^i}, i \geq -k\}$ . Then if  $\alpha \in \mathbb{C}_\infty$  is a zero of  $L_N(t)$  and  $\alpha \notin S$ ,  $\alpha$  must be a zero of  $B_N(t, \theta)$ . Observe that by Proposition 4.5 and Proposition 4.6, we have:

$$\deg_t \Lambda_r(N) = \deg_t B_N(t, \theta).$$

By Proposition 4.5, the zeros in  $\mathbb{C}_\infty$  of  $\Lambda_r(N)$  are not in  $S$ . By Lemma 2.1 and Lemma 3.3,  $\theta^{-r}B_N(t, \theta)$  and  $\Lambda_r(t)$  have the same Newton polygon. Thus, by the proof of Proposition 4.5 and the properties of Newton polygons ([13], chapter 2), we get in  $K_\infty[t]$ :

$$B_N(t, \theta) = \lambda \prod_{j=1}^r P_j(t),$$

where  $\lambda \in \mathbb{F}_p^\times$ ,  $P_j(t)$  is an irreducible monic element in  $K_\infty[t]$ ,  $P_j(t) \neq P_{j'}(t)$  for  $j \neq j'$ . Furthermore each root of  $P_j(t)$  generates a totally ramified extension of  $K_\infty$  and  $p^{k+1} > \deg_t P_j(t) > n_k p^k$ . Also note that  $\deg_t P_j(t) \not\equiv 0 \pmod{p^k}$  and  $\deg_t P_j(t) \equiv 1 \pmod{p-1}$ . Observe that if  $x \in \cup_{i \in \mathbb{Z}} (\mathbb{F}_p((\frac{1}{\theta^{p^i}}))^{ab})^{perf}$ , then there exist  $l \geq 0, m \in \mathbb{Z}, d \geq 1, p \equiv 1 \pmod{d}$ , such that  $v_\infty(x) = \frac{m}{dp^l}$ . Thus,  $P_j(t)$  has no roots in  $\cup_{i \in \mathbb{Z}} (\mathbb{F}_p((\frac{1}{\theta^{p^i}}))^{ab})^{perf}$ .

Write:

$$\theta^{-r}B_N(t, \theta) = \sum_{j=0}^r \beta_j(t) \theta^{-j}, \beta_j(t) \in \mathbb{F}_p[t].$$

Observe that  $\beta_0(t) = 1$  and by the above discussion,  $\theta^{-r}B_N(t, \theta)$  and  $\Lambda_r(t)$  have the same Newton polygon (as polynomials in  $t$ ). Now, by Proposition 4.5 and Proposition 4.6, we get:

$$\deg_t \beta_r(t) = \deg_t \alpha_r(t).$$

We deduce that:

$$i = 0, \dots, r, \deg_t \beta_i(t) = \deg_t \alpha_i(t).$$

By the proof of Proposition 4.5, for  $i \in \{1, \dots, r-1\}$ ,  $\deg_t \beta_{i+1}(t) - \deg_t \beta_i(t) < \deg_t \beta_i(t) - \deg_t \beta_{i-1}(t)$ . Thus the edges of the Newton polygon of  $\theta^{-r}B_N(t, \theta)$  viewed as polynomial in  $\frac{1}{\theta}$  are  $(i, -\deg_t(\beta_i(t))), i = 0, \dots, r$ .  $\square$

## 5. SOME HINTS FOR PROBLEM 1 FOR GENERAL $q$ .

In this section  $q$  is no longer assumed to be equal to  $p$ .

### 5.1. The work of J. Sheats.

For  $N, d \geq 1$ , we set  $U_d(N) = U_d(N, (q-1, \dots, q-1))$ . Thus:

$$S_d(N) := \sum_{a \in A_{+,d}} a(t)^N = (-1)^d \sum_{\mathbf{m} \in U_d(N)} C(N, \mathbf{m}) t^{\deg \mathbf{m}}.$$

J. Sheats proved ([18], Lemma 1.3) that if  $U_d(N) \neq \emptyset$ ,  $U_d(N)$  has a unique optimal element and it is the greedy element of  $U_d(N)$ . In particular  $U_d(N) \neq \emptyset \Leftrightarrow S_d(N) \neq 0$ . Observe that if  $\mathbf{m} = (m_0, \dots, m_d) \in U_d(N)$ , then  $(m_0, \dots, m_{d-2}, m_{d-1} + m_d) \in U_{d-1}(N)$ . In particular  $U_d(N) \neq \emptyset \Rightarrow U_{d-1}(N) \neq \emptyset$ .

#### Proposition 5.1.

1) Let  $d \geq 1$  such that  $U_d(N) \neq \emptyset$ . Then, for  $j \in \{1, \dots, d-1\}$ , we have:

$$\deg_t S_j(N) - \deg_t S_{j-1}(N) > \deg_t S_{j+1}(N) - \deg_t S_j(N).$$

2) Let  $d \geq 1$  such that  $U_{d+1}(N) \neq \emptyset$ . Let  $\mathbf{m}$  be the greedy element of  $U_{d+1}(N)$ . Then:

$$\deg_t S_d(N) > N(d-1),$$

and:

$$\deg_t S_d(N) - \deg_t S_{d-1}(N) > m_{d+1}.$$

*Proof.*

1) Observe that this assertion is a consequence of the proof of [18], Theorem 1.1 (see pages 127 and 128 of [18]).

2) Let  $\mathbf{m} = (m_0, \dots, m_{d+1})$  be the greedy element of  $U_{d+1}(N)$ . Define  $\mathbf{m}' = (m_0, \dots, m_d) \in U_d(N - m_{d+1})$ . Then:

$$m_d \equiv 0 \pmod{q-1}, m_d \geq q-1.$$

Furhermore, observe that  $\mathbf{m}'$  is the greedy element of  $U_d(N - m_{d+1})$ . By [18], Lemma 1.3 and Proposition 4.6, we get:

$$\deg_t S_d(N - m_{d+1}) > (N - m_{d+1})(d-1).$$

Let  $\mathbf{m}'' = (m_0, \dots, m_{d-1}, m_d + m_{d+1}) \in U_d(N)$ . We have:

$$\deg_t S_d(N) \geq m_1 + \dots + (d-1)(m_{d-1}) + d(m_d + m_{d+1}).$$

Thus:

$$\begin{aligned} \deg_t S_d(N) &\geq \deg_t S_d(N - m_{d+1}) + dm_{d+1} \\ &> (N - m_{d+1})(d-1) + dm_{d+1} = (d-1)N + m_{d+1}. \end{aligned}$$

Thus:

$$\begin{aligned} \deg_t S_d(N) &> N(d-1), \\ \deg_t S_d(N) - \deg_t S_{d-1}(N) &\geq \deg_t S_d(N) - (d-1)N > m_{d+1}. \end{aligned}$$

□

To conclude this paragraph, we recall the following crucial result due to G. Böckle ([8], Theorem 1.2):

$$S_d(N) \neq 0 \Leftrightarrow d(q-1) \leq \text{Min}\{\ell_q(p^i N), i \in \mathbb{N}\}.$$

An integer  $N \geq 1$  will be called  $q$ -minimal if:

$$\left\lfloor \frac{\ell_q(N)}{q-1} \right\rfloor = \text{Min}\left\{\left\lfloor \frac{\ell_q(p^i N)}{q-1} \right\rfloor, i \in \mathbb{N}\right\}.$$

## 5.2. Consequences of Sheats results.

Let  $N \geq 1$ , and write:

$$L_N(t) = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \alpha_{i,N}(t) \in \mathbb{F}_q[t].$$

**Proposition 5.2.** *Let  $d \geq 1$  such that  $U_{d+1}(N) \neq \emptyset$ . Set:*

$$\Lambda_d(t) = \sum_{i=0}^d \alpha_{i,N}(t) \theta^{-i} \in K_\infty[t].$$

*Then  $\deg_t \Lambda_d(t) = \deg_t S_d(N)$  and the edge points of the Newton polygon of  $\Lambda_d(t)$  are:*

$$(\deg_t S_i(N), i), i = 0, \dots, d.$$

*Proof.* The proof uses similar arguments as that used in the proof of Proposition 4.5. Let  $j \geq 0$ , then (see Lemma 3.1), we have:

$$\alpha_{j,N}(t) = \sum_{\ell(\mathbf{k}) + w(\mathbf{k}) = j} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}.$$

Observe that:

$$\deg_t \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \leq \ell(\mathbf{k})N.$$

Thus, for  $j = 0, \dots, d$ , by Proposition 5.1, assertion 2), we get:

$$\deg_t \alpha_{j,N}(t) = \deg_t S_j(N).$$

In particular, again by Proposition 5.1, assertion 2), we have:

$$\deg_t \Lambda_d(t) = \deg_t S_d(N).$$

Finally, by Proposition 5.1, assertion 1),  $(\deg_t S_i(N), i)$ ,  $i = 0, \dots, d$ , are the edge points of the Newton polygon of  $\Lambda_d(t)$ . □

**Lemma 5.3.** *We assume that  $N$  is  $q$ -minimal,  $N \equiv 1 \pmod{q-1}$ . We also assume that  $r \geq 1$  (recall that  $r = \frac{\ell_q(N)-q}{q-1}$ ).*

1) *Write  $N = \sum_{l=0}^k n_l q^l$ ,  $n_0, \dots, n_k \in \{0, \dots, q-1\}$ ,  $n_k \neq 0$ . Then there exists an integer  $0 \leq m \leq k$  such that  $n_m \not\equiv 0 \pmod{p}$ .*

2) *Let  $n = \text{Max}\{l, 0 \leq l \leq k, n_l \not\equiv 0 \pmod{p}\}$ . Let  $\mathbf{m} = (m_0, \dots, m_{r+1}) \in U_{r+1}(N)$  be the greedy element, then:*

$$m_{r+1} = q^n.$$

3) *Assume that  $n < k$ . Then, for  $i \in \{1, \dots, r-1\}$ , we have:*

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > pq^k.$$

*Proof.*

1) Let's assume that the assertion is false. Then:

$$\ell_q\left(\frac{q}{p}N\right) = \frac{\ell_q(N)}{p}.$$

This contradicts the  $q$ -minimality of  $N$ .

2) Observe that:

$$\forall i \geq 0, \ell_q(p^i N) = \ell_q(p^i N - p^i q^n) + \ell_q(p^i).$$

Therefore,  $N - q^n$  is  $q$ -minimal. Thus, by Böckle's result:  $U_{r+1}(N - q^n) \neq \emptyset$ . This easily implies that there exists  $\mathbf{n} = (n_0, \dots, n_{r+1}) \in U_{r+1}(N)$  such that:

$$n_{r+1} = q^n.$$

We have:

$$1 + (q-1)(r+1) = \ell_q(N) = \sum_{i=0}^{r+1} \ell_q(m_i).$$

Furthermore:

$$\sum_{i=0}^r \ell_q(m_i) \equiv 0 \pmod{q-1},$$

and

$$\sum_{i=0}^r \ell_q(m_i) \geq (q-1)(r+1).$$

Thus:

$$\ell_q(m_{r+1}) = 1.$$



This implies that  $m_{r+1}$  is a power of  $q$  and since  $\mathbf{m}$  is the greedy element of  $U_{r+1}(N)$ , we also have:  $m_{r+1} \geq q^n$ . Since there is no carryover  $p$ -digits in the sum  $m_0 + \dots + m_{r+1}$ , by the definition of  $n$ , we deduce that  $m_{r+1} = q^n$ .

3) Let  $\mathbf{m}' = (m_0, \dots, m_{r-1}, m_r + m_{r+1}) \in U_r(N)$ . If  $\mathbf{n}$  is the greedy element of  $U_r(N)$  then:

$$n_r \geq m_r + m_{r+1}.$$

Since  $\mathbf{m}$  is the greedy element of  $U_{r+1}(N)$ , we have:

$$m_0 \leq m_1 \leq \dots \leq m_r.$$

Since there is no carryover  $p$ -digits in the sum  $m_0 + \dots + m_{r+1}$ , and  $n < k$ , this implies that:

$$m_r = \sum_{l=0}^k j_l q^l, j_l \in \{0, \dots, q-1\}, j_k \neq 0, j_k \equiv 0 \pmod{p}.$$

Thus:

$$m_r > p q^k.$$

It remains to apply Proposition 5.1. □

### 5.3. Zeros of $B_N(\theta, t)$ .

The following theorem implies in particular Theorem 1.1 in the case where  $N$  is  $q$ -minimal.

**Theorem 5.4.** *We assume that  $N$  is  $q$ -minimal,  $N \equiv 1 \pmod{q-1}$ . We also assume that  $r \geq 1$ .*

1)  $B_N(\theta, \theta) \neq 0$  and the zeros of  $B_N(t, \theta)$  are algebraic integers (i.e. they are integral over  $A$ ). Furthermore:  $(r-1)N < \deg_t B_N(t, \theta) < rN$ .

2)  $B_N(\theta, t)$  has only simple roots and its roots belong to  $\mathbb{F}_p((\frac{1}{\theta})) \setminus \{\theta\}$ .

*Proof.* The proof is in the same spirit as that of the proofs of Proposition 4.6 and Theorem 4.7.

Write  $N = \sum_{l=0}^k n_l q^l$ , where  $n_0, \dots, n_k \in \{0, \dots, q-1\}$ , and  $n_k \neq 0$ . Recall that  $r+1 = \lceil \frac{\ell_q(N)}{q-1} \rceil$ . Let  $n \geq 0$  be the integer such that  $n = \text{Max}\{l, n_l \not\equiv 0 \pmod{p}\}$  (see Lemma 5.3). Let  $\varepsilon_N(t) \in \mathbb{F}_p[t, \frac{1}{\theta}]$  be the polynomial determined by the congruence:

$$\varepsilon_N(t) \equiv \Lambda_r(t) \prod_{j=0}^k \prod_{i \geq 0} (1 - \frac{t q^j}{\theta q^i})^{n_j} \pmod{\frac{1}{\theta^{r+1}} \mathbb{F}_p[t][[\frac{1}{\theta}]]},$$

where  $\Lambda_r(t) = \sum_{l=0}^r \alpha_{l,N}(t) \theta^{-l}$ . We can write:

$$\varepsilon_N(t) = \sum_{l=0}^r \eta_l(t) \theta^{-l}, \eta_l(t) \in \mathbb{F}_p[t].$$

Note that  $\eta_l(t)$  is a  $\mathbb{F}_p$ -linear combination of terms of the form

$$x_{l,j,u} = \alpha_{l-u,N}(t) \theta^{-l+u} t^j \theta^{-u}, \quad j \leq q^k u.$$

By Proposition 5.2, we have:

$$l = 0, \dots, r, \deg_t \alpha_{l,N}(t) = \deg_t S_l(N).$$

1) Case  $n = k$ .

By Proposition 5.1 and Lemma 5.3 :

$$l = 0, \dots, h-1, \deg_t \alpha_{h-l,N}(t) < \deg_t \alpha_{h,N}(t) - q^k l.$$

Thus, if  $l \neq h$  or  $u \neq 0$ , we get:

$$l = 0, \dots, h, \deg_t x_{l,j,u} < \deg_t \alpha_{l,N}(t).$$

Therefore:

$$l = 0, \dots, r, \deg_t \eta_l(t) = \deg_t S_l(N).$$

2) Case  $n < k$ .

As in the proof of the case  $n = k$ , we get by Proposition 5.1 and Lemma 5.3:

$$l = 0, \dots, r-1, \deg_t \eta_l(t) = \deg_t S_l(N).$$

Furthermore, by Proposition 5.1 and Lemma 5.3, we have for  $l \geq 2$ :

$$\begin{aligned} \deg_t S_{r-l}(N) &< \deg_t S_{r-1}(N) - pq^k(l-1), \\ \deg_t S_r(N) - \deg_t S_{r-1}(N) &> q^n. \end{aligned}$$

Thus, for  $u \geq 2$ :

$$\deg_t x_{r,j,u} \leq \deg_t S_{r-u}(N) + q^k u < \deg_t S_r(N).$$

Now, observe that:

$$\prod_{j=0}^k \prod_{i \geq 0} (1 - \frac{t^{q^j}}{\theta^{q^i}})^{n_j} \equiv 1 - \frac{1}{\theta} \sum_{l=0}^k n_l t^{q^l} \pmod{\frac{1}{\theta^2} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

Thus:

$$\deg_t x_{r,j,1} \leq \deg_t S_{r-1}(N) + q^n < \deg_t S_r(N).$$

Thus we get:

$$\deg_t \eta_r(t) = \deg_t S_r(N).$$

Now, observe that:

$$\frac{(-1)^{\frac{\ell_q(N)-1}{q-1}} \theta^{-r} B_N(t, \theta)}{\prod_{j \geq 1} (1 - \theta^{1-q^j})} \equiv \varepsilon_N \pmod{\frac{1}{\theta^{r+1}} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

We easily deduce that:

$$\theta^{-r} B_N(t, \theta) = \sum_{l=0}^r \beta_l(t) \theta^{-l}, \beta_l(t) \in \mathbb{F}_p[t], \deg_t \beta_l(t) = \deg_t S_l(N), l = 0, \dots, r.$$

Observe that, by Proposition 5.1, we have  $\deg_t S_r(N) > N(r-1)$ , and it is obvious that  $\deg_t S_r(N) < rN$ . Now, we get assertion 1) and 2) by the same reasoning as that used in the proof of Theorem 4.7.  $\square$

**Corollary 5.5.** *We assume that  $N$  is  $q$ -minimal,  $N \equiv 1 \pmod{q-1}$ . We also assume that  $r \geq 1$ . Then,  $B_N(t, \theta)$  has at most one zero in  $\{\theta^{q^i}, i \in \mathbb{Z}\}$ .*

*Proof.* Let's assume that  $B_N(t, \theta)$  has a zero  $\alpha \in \{\theta^{q^i}, i \in \mathbb{Z}\}$ . Let  $n$  be the integer introduced in the proof of Theorem 5.4. Then:

$$\alpha = \theta^{q^{-i}}, k \geq i > n,$$

where  $q^k \leq N < q^{k+1}$ . Thus  $n < k$ , and therefore, by Lemma 5.3, we must have:

$$\deg_t S_r(N) - \deg_t S_{r-1}(N) = q^i.$$

By the proof of Theorem 5.4, we have:

$$\theta^{-r} B_N(t, \theta) = \left( \frac{t^{q^i}}{\theta} - 1 \right) F(t),$$

where  $F(t) = \sum_{l=0}^{r-1} \nu_l(t)\theta^{-l}$ ,  $\nu_l(t) \in \mathbb{F}_p[t]$ ,  $\deg_t \nu_l(t) = \deg_t S_l(N)$ ,  $l = 0, \dots, r-1$ . Furthermore  $\nu_0(t) = -1$ . This implies that the zeros of  $F(t)$  are not in  $\{\theta^{q^i}, i \in \mathbb{Z}\}$ .  $\square$

**Corollary 5.6.**

- 1) The polynomial  $\mathbb{B}_s$  is square-free, i.e.  $\mathbb{B}_s$  is not divisible by the square of a non-trivial polynomial in  $\mathbb{F}_q[t_1, \dots, t_s, \theta]$ .
- 2) For all  $l, n \in \mathbb{N}$ ,  $\mathbb{B}_s$  is relatively prime to  $(t_1^l - \theta^{q^n})$  (observe that  $\mathbb{B}_s$  is a symmetric polynomial in  $t_1, \dots, t_s$ ).
- 3) For all monic irreducible prime  $P$  of  $A$ ,  $\mathbb{B}_s$  is relatively prime to  $P(t_1) \cdots P(t_s) - P$ .

*Proof.* Let  $N = q^{e_1} + \dots + q^{e_s}$ ,  $0 \leq e_1 < e_2 < \dots < e_s$ . Then:

$$B_N(t, \theta) = \mathbb{B}_s|_{t_i = t^{q^{e_i}}}.$$

We observe that  $N$  is  $q$ -minimal. Thus we can apply Theorem 5.4. This Theorem and its proof imply that  $B_N(t, \theta)$  is square-free and has no roots in  $\{\theta^{q^i}, i \in \mathbb{Z}\}$ . This proves 1) and 2).

Let  $P$  be a monic irreducible polynomial in  $A$ . Suppose that  $P(t_1) \cdots P(t_s) - P$  and  $\mathbb{B}_s$  are not relatively prime. Then  $P(t)^N - P$  and  $B_N(t, \theta)$  are not relatively prime. But, by the proof of Theorem 5.4, if  $\alpha \in \mathbb{C}_\infty$  is a root of  $B_N(t, \theta)$ , then:

$$v_\infty(\alpha) > \frac{-1}{N}.$$

Now, observe that if  $\beta \in \mathbb{C}_\infty$  is a root of  $P(t)^N - P$ , then  $v_\infty(\beta) = \frac{-1}{N}$ . This leads to a contradiction.  $\square$

Note that assertion 1) of the above Corollary gives the cyclicity result implied by [7], Theorem 3.17, but by a completely different method.

## 6. AN EXAMPLE

We study here an example of an  $N$  which is not  $q$ -minimal, so that our method does not apply. We choose  $q = 4$ , and  $N = 682 = 2 + 2 \times 4 + 2 \times 4^2 + 2 \times 4^3 + 2 \times 4^4$ . We get  $l_q(N) = 10 = 3q - 2$  so that  $\deg_\theta(B_N(t, \theta)) = 2$ . Moreover,  $l_q(pN) = 5$  so that  $N$  is not  $q$ -minimal. By using the table of section B, we get :

$$B_N(t, \theta) = \theta^2 + \theta(t^{10} + t^{34} + t^{40} + t^{130} + t^{136} + t^{160} + t^{514} + t^{520} + t^{544} + t^{640}) \\ + (t^{170} + t^{554} + t^{650} + t^{674} + t^{680}).$$

The Newton polygon of  $B_N(t, \theta)$  has then the end points  $(0, -2)$ ,  $(640, -1)$ ,  $(680, 0)$ . We deduce that  $B_N(t, \theta)$  has 640 distinct zeroes of valuation  $-\frac{1}{640}$  and 40 distinct zeros of valuation  $-\frac{1}{40}$ . Similarly,  $B_N(\theta, t)$  has two zeros of respective valuations  $-40$  and  $-640$ . In particular, we still have that  $B_N(t, \theta)$  has no zero of the form  $\theta^{q^i}$ ,  $i \in \mathbb{Z}$ , and an affirmative answer to Problem 1.

## A. APPENDIX: THE DIGIT PRINCIPLE AND DERIVATIVES OF CERTAIN $L$ -SERIES, BY B. ANGLÈS, D. GOSS, F. PELLARIN AND F. TAVARES RIBEIRO

We keep the notation of the article.

Let  $N$  be a positive integer. We consider its base- $q$  expansion  $N = \sum_{i=0}^r n_i q^i$ , with  $n_i \in \{0, \dots, q-1\}$ . We recall that  $\ell_q(N) = \sum_{i=0}^r n_i$  and the definition of the Carlitz factorial :

$$\Pi(N) = \prod_{i \geq 0} D_i^{n_i} \in A^+,$$

where  $[k] = \theta^{q^k} - \theta$  if  $k > 0$  and  $D_j = [j][j-1] \cdots [1]^{q^{j-1}}$  for  $j > 0$ .

It is easy to see (the details are in §A.1, A.2 and A.3) that, if we denote by  $a'$  the derivative  $\frac{d}{d\theta}a$  of  $a \in A$  with respect to  $\theta$ , the series

$$\sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^N}{a}$$

converges in  $K_\infty$  to a limit that we denote by  $\delta_N$ . In particular, if  $n = q^j$  with  $j > 0$ , we will see (Proposition A.4) that

$$\delta_1 = -\sum_{k \geq 1} \frac{1}{[k]} \quad \text{and} \quad \delta_{q^j} = \frac{D_j}{[j]} \tilde{\pi}^{1-q^j}.$$

Our aim is to prove the following:

**Theorem A.1.** *If  $N \geq q$  is such that  $N \equiv 1 \pmod{q-1}$  and  $\ell_q(N) \geq q$ , then*

$$\frac{\delta_N}{\tilde{\pi}} = \beta_N \frac{\Pi(N)}{\Pi(\lfloor \frac{N}{q} \rfloor)^q} \prod_{k=1}^r \left( \frac{\delta_{q^k}}{\tilde{\pi}} \right)^{n_k},$$

where for  $x \in \mathbb{R}$ ,  $[x]$  denotes the integer part of  $x$ , and where  $\beta_N = (-1)^{\frac{s-1}{q-1}} B_N(\theta, \theta)$ .

Our Theorem A.1 can be viewed as a kind of *digit principle* for the values  $\delta_j$  in the sense of [10].

The plan of this appendix is the following. In §A.1, we recall the first properties of Anderson and Thakur function  $\omega$ . In §A.2 we discuss the one-digit case of our Theorem, while the general case is discussed in §A.3.

**A.1. The Anderson-Thakur function.** Recall that  $\mathbb{T}_t$  denotes the Tate algebra over  $\mathbb{C}_\infty$  in the variable  $t$ ,  $C : A \rightarrow A\{\tau\}$  is the Carlitz module ([13], chapter 3), in other words,  $C$  is the morphism of  $\mathbb{F}_q$ -algebras given by  $C_\theta = \tau + \theta$ , and

$$\exp_C = \sum_{i \geq 0} \frac{1}{D_i} \tau^i \in \mathbb{T}_t\{\{\tau\}\}$$

is the Carlitz exponential. In particular, we have the following equality in  $\mathbb{T}_t\{\{\tau\}\}$  :

$$\exp_C \theta = C_\theta \exp_C.$$

Let us choose a  $(q-1)$ -th root  ${}^{q-1}\sqrt{-\theta}$  of  $-\theta$  in  $\mathbb{C}_\infty$  and set:

$$\tilde{\pi} = \theta {}^{q-1}\sqrt{-\theta} \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in \mathbb{C}_\infty^\times.$$

We recall the Anderson-Thakur function ([1], proof of Lemma 2.5.4):

$$\omega(t) = {}^{q-1}\sqrt{-\theta} \prod_{j \geq 0} \left( 1 - \frac{t}{\theta q^j} \right)^{-1} \in \mathbb{T}_t^\times.$$

To give an idea of how to compute  $\exp_C(f)$  for certain  $f$  in  $\mathbb{T}_t$ , we verify here that

$$\exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right) = \sum_{j \geq 0} \frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j}-t)}$$

is a well defined element of  $\mathbb{T}_t$ . Indeed, for  $j \geq 0$  :

$$v_\infty\left(\frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j}-t)}\right) = q^j \left(j+1 - \frac{q}{q-1}\right).$$

Therefore  $\sum_{j \geq 0} \frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j}-t)}$  converges in  $\mathbb{T}_t$ .

We will need the following crucial result in the sequel:

**Proposition A.2.** *We have the following equality in  $\mathbb{T}_t$  :*

$$\omega(t) = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

*Proof.* It is a consequence of the formulas established in [15]. We give details for the convenience of the reader. Let us set

$$F(t) = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

We observe that:

$$\begin{aligned} C_\theta(F(t)) &= \exp_C\left(\frac{\theta\tilde{\pi}}{\theta-t}\right) = \exp_C\left(\frac{(\theta-t+t)\tilde{\pi}}{\theta-t}\right) \\ &= \exp_C(\tilde{\pi}) + \exp_C\left(\frac{t\tilde{\pi}}{\theta-t}\right) = t \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right) = tF(t). \end{aligned}$$

Therefore:

$$\tau(F(t)) = (t-\theta)F(t).$$

But we also have:

$$\tau(\omega(t)) = (t-\theta)\omega(t).$$

Finally, we get:

$$\tau\left(\frac{F(t)}{\omega(t)}\right) = \frac{F(t)}{\omega(t)}.$$

It is a simple and well-known consequence of a ultrametric variant of Weierstrass preparation Theorem that  $\{f \in \mathbb{T}_t, \tau(f) = f\} = \mathbb{F}_q[t]$ . Since  $\omega \in \mathbb{T}_t^\times$ , we have then:

$$\frac{F(t)}{\omega(t)} \in \mathbb{F}_q[t].$$

Now observe that

$$F(t) = \exp_C\left(\sum_{j \geq 0} \frac{\tilde{\pi}}{\theta^{j+1}} t^j\right) = \sum_{j \geq 0} \lambda_{\theta^{j+1}} t^j$$

and that, for all  $j \geq 0$ ,  $v_\infty(\lambda_{\theta^{j+1}}) = j+1 - \frac{q}{q-1}$ . This implies  $v_\infty(\frac{F(t)}{\lambda_\theta} - 1) > 0$ . By the definition of  $\omega(t)$ , we also have  $v_\infty(\frac{\omega(t)}{\lambda_\theta} - 1) > 0$ . Thus:

$$v_\infty\left(\frac{F(t)}{\omega(t)} - 1\right) > 0.$$

Since  $\frac{F(t)}{\omega(t)} \in \mathbb{F}_q[t]$ , we get  $\omega(t) = F(t)$ . □

Notice that  $\omega(t)$  defines a meromorphic function on  $\mathbb{C}_\infty$  without zeroes. Its only poles, simple, are located at  $t = \theta, \theta^q, \theta^{q^2}, \dots$ . As a consequence of Proposition A.2, we get:

**Corollary A.3.** *Let  $j \geq 0$  be an integer, then:*

$$(t - \theta^{q^j})\omega(t) \big|_{t=\theta^{q^j}} = -\frac{\tilde{\pi}^{q^j}}{D_j}.$$

**A.2. The one digit case.** Let us consider the following  $L$ -series:

$$L(t) = L_1(t) = \sum_{k \geq 0} \sum_{a \in A_{+,k}} \frac{a(t)}{a} \in \mathbb{T}_t.$$

By Proposition 3.7, we have the following equality in  $\mathbb{T}_t$  (see [15], Theorem 1):

$$\frac{L(t)\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

This implies that  $L(t)$  extends to an entire function on  $\mathbb{C}_\infty$  (see also Lemma 2.1 or [3, Proposition 6]). We set:

$$L'(t) = \sum_{k \geq 0} \sum_{a \in A_{+,k}} \frac{a'(t)}{a} \in \mathbb{T}_t,$$

where  $a'(t)$  denotes the derivative  $\frac{d}{dt}a(t)$  of  $a(t)$  with respect to  $t$ . The derivative  $\frac{d}{dt}$  inducing a continuous endomorphism of the algebra of entire functions  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ ,  $L'(t)$  extends to an entire function on  $\mathbb{C}_\infty$ . Thus, for  $j \geq 0$  an integer,  $\sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^{q^j}}{a}$  converges in  $K_\infty$  and we have:

$$\delta_{q^j} = \sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^{q^j}}{a} = L'(t) \big|_{t=\theta^{q^j}}.$$

**Proposition A.4.** *The following properties hold:*

(1) *We have:*

$$\delta_1 = -\sum_{k \geq 1} \frac{1}{[k]}.$$

(2) *Let  $j \geq 1$  be an integer, then:*

$$\delta_{q^j} = \frac{\Pi(q^j)}{[j]} \tilde{\pi}^{1-q^j}.$$

*Proof.* (1) It is well known that, for  $n > 0$ ,  $D_n = \prod_{a \in A_{+,n}} a$ . Therefore,

$$\sum_{a \in A_{+,n}} \frac{a'}{a} = -\frac{1}{[n]} \text{ from which the first formula follows.}$$

(2) By [13, Remark 8.13.10], we have:

$$L(t) \big|_{t=\theta^{q^j}} = 0.$$

Thus:

$$\delta_{q^j} = L'(t) \big|_{t=\theta^{q^j}} = \frac{L(t)}{t - \theta^{q^j}} \big|_{t=\theta^{q^j}}.$$

But,

$$\frac{\frac{L(t)}{t - \theta^{q^j}}(t - \theta^{q^j})\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

It remains to apply Corollary A.3. □

**Remark A.5.** The transcendence over  $K$  of the “bracket series”  $\delta_1 = \sum_{i \geq 1} \frac{1}{[i]}$  was first obtained by Wade [23]. The transcendence of  $\delta_1$  directly implies the transcendence of  $\tilde{\pi}$ .

**A.3. The several digits case.** As a consequence of [5], Lemma 7.6 (see also [3], Corollary 21), the series  $L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a}$  has a zero of order at least  $N$  at  $t = \theta$ . Thus,

$$\tilde{L}_N(t) = \sum_{d \geq 1} \sum_{a \in A_{+,d}} \frac{a'(t)^N}{a}$$

defines an entire function on  $\mathbb{C}_\infty$  such that

$$\delta_N = \tilde{L}_N(\theta).$$

*Proof of Theorem A.1.* Recall that  $N = \sum_{i=0}^r n_i q^i$ , is the  $q$ -expansion of  $N$ . We set  $s = \ell_q(N)$ . We can assume that  $s \geq q$  by Proposition A.4. From the definition of  $B_N(t, \theta)$  in §2.3, we have :

$$(-1)^{\frac{s-1}{q-1}} B_N(t, \theta) = L_N(t) \left( \prod_{i=0}^r \omega(t^{q^i})^{n_i} \right) \tilde{\pi}^{-1} \in A[t].$$

Since

$$\delta_N = \tilde{L}_N(\theta) = \left( \frac{L_N(t)}{\prod_{i=0}^r (t - \theta^{q^i})^{n_i}} \right)_{|t=\theta},$$

we obtain, by Corollary A.3 and our previous discussions:

$$\beta_N = \frac{\delta_N \prod_{i=0}^r \left( \frac{-\tilde{\pi}^{q^i}}{D_i} \right)^{n_i}}{\tilde{\pi}}.$$

Now, by Proposition A.4, we have, for all  $k \geq 1$ ,  $D_k = [k] \delta_{q^k} \tilde{\pi}^{q^k-1}$ . We obtain the Theorem by using the fact that:

$$\frac{\Pi(N)}{\Pi\left(\left[\frac{N}{q}\right]^q\right)} = \prod_{k \geq 1} [k]^{n_k}.$$

□

## B. TABLE

We give an explicit expression of the polynomials  $\mathbb{B}_s$  for  $s \in \{q, 2q-1, 3q-2\}$ . We recall that  $\mathbb{B}_s$  is monic of degree  $r = \frac{s-q}{q-1}$ . One obtains the corresponding

expressions for  $B_N(t, \theta)$  if  $\ell_q(N) = s$  by evaluating the variables  $t_i$ 's as in §2.3.

$$\begin{aligned} \mathbb{B}_q &= 1, \\ \mathbb{B}_{2q-1} &= \theta - \sum_{i_1 < \dots < i_q} t_{i_1} \cdots t_{i_q}, \\ \mathbb{B}_{3q-2} &= \theta^2 - \theta \left( \sum_{i_1 < \dots < i_{2q-1}} \prod_{j=1}^{2q-1} t_{i_j} + \sum_{i_1 < \dots < i_q} \prod_{j=1}^q t_{i_j} \right) + \\ &\quad + \left( \sum_{i_1 < \dots < i_q} \prod_{j=1}^q t_{i_j}^2 + \sum_{m_1 < \dots < m_{q-1}, m_j \neq i_j, j=1}^{q-1} \prod_{j=1}^{q-1} t_{i_j} + \sum_{i_1 < \dots < i_{2q}} \prod_{j=1}^{2q} t_{i_j} \right). \end{aligned}$$

One easily computes the discriminant of  $\mathbb{B}_{3q-2}$  from this table. It is then an easy computation to prove that  $B_N(\theta, t)$  has only simple roots for all  $N$  such that  $\ell_q(N) = 3q - 2$ .

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